Dasgupta-Ma Renormalization in the Random Heisenberg XX model and the Random Singlet Phase

Mauricio Fonseca Fernández

(Dated: 10 May 2018)

Random spin chain models are obtained by introducing quenched disorder in the interaction parameters of a given spin chain. A renormalization group approach developed by Dasgupta and Ma proves to be useful in studying this kind of systems. In this digest, this renormalization group method is explained in the context of the random Heisenberg XX model, which leads to a renormalization flow equation for the distribution of interaction strengths. The derivation of a special fixed point of this equation is provided and some properties of this fixed point, known as the random singlet phase, are discussed.

I. INTRODUCTION

Spins in a lattice models are extensively studied because they provide important insight into prototypical phenomena despite the simplicity of their formulation.

Consider the one dimensional quantum Heisenberg XYZ antiferromagnetic chain defined by the Hamiltonian

$$H = \sum_{r=1}^{L-1} K_r^\alpha S_r^\alpha S_{r+1}^\alpha \tag{I.1}$$

where the implicitly summed α represents the Cartesian coordinates X, Y, Z. $\vec{S_r}$ are spin-$\frac{1}{2}$ operators and the coupling constants $\vec{K}_r$ are fixed in time but randomly distributed in space (a condition known as *quenched randomness*) according to some distributions $P^x(K^x)$, with $0 \leq K^x \leq J^x$ (and similar for y and z).

This Hamiltonian describes a general system which can be modified into other widely studied models by imposing conditions on its parameters. We focus in the one dimensional quantum Heisenberg XX antiferromagnetic chain obtained by making all $K_n^z = 0$ and $K_n^x = K_n^y$. Here we need only to consider one distribution $P^x(K^x) = P(K)$, with $0 \leq K \leq J$. The Hamiltonian now is given by

$$H = \sum_{r=1}^{L-1} K_r \vec{S}_r \cdot \vec{S}_{r+1} \tag{I.2}$$

with the $\vec{S}_r$ in the XY plane*.

The notion of phases and phase transitions is always of interest when studying many body systems. Here we are interested in exposing a special phase that can be observed in many random spin chain models for the specific case of the system described in (I.2). To do so, we will use a real space renormalization group approach known as Dasgupta-Ma renormalization (also strong disorder renormalization group).

*It is important to remember that $\vec{S}$ still has a Z component and we still use the basis of eigenstates of S^z. The reason we regard these $\vec{S}_r$ as living in the XY plane is to simplify the notation, given that the $\vec{S}_r^z$ do not appear in the Hamiltonian because their coefficients K^z are all zero.

II. DASGUPTA-MA RENORMALIZATION

Dasgupta and Ma[1] developed a renormalization scheme that allows to study the properties of the random Heisenberg antiferromagnet Hamiltonian defined in (I.2). We start by taking the strongest bond between neighboring spins in the chain, labeled by J according to the bond distribution, and labeling the spins it connects $\vec{S}_1$ and $\vec{S}_2$ (see figure 1). For the moment we ignore the other neighbors of these, $\vec{S}_1'$ and $\vec{S}_2'$. The local Hamiltonian considering only these two spins is

$$H_0 = J\vec{S}_1 \cdot \vec{S}_2 \tag{II.1}$$

This is a simple quantum mechanical system of two interacting spins that has the singlet state $|s\rangle$ as its ground state and the three triplet states $|t_i\rangle$ as its excited states. Remember that in the usual basis of eigenstates of S_i^z

$$\begin{aligned} |s\rangle &= \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle) \\ |t_1\rangle &= |\uparrow\uparrow\rangle \\ |t_0\rangle &= \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle) \\ |t_{-1}\rangle &= |\downarrow\downarrow\rangle \end{aligned}$$

Writing

$$\begin{aligned} S^x &= \frac{1}{2}(S^+ + S^-) \\ S^y &= \frac{1}{2i}(S^+ - S^-) \end{aligned} \tag{II.2}$$

one can express $\vec{S}_1 \cdot \vec{S}_2$ explicitly as

$$\vec{S}_1 \cdot \vec{S}_2 = \frac{1}{2}(S_1^+ S_2^- + S_1^- S_2^+) \tag{II.3}$$

And with this it is easy to find that

$$\begin{aligned} E_s &= -\frac{1}{2}J \\ E_{t_1} &= E_{t_{-1}} = 0 \\ E_0 &= \frac{1}{2}J \end{aligned} \tag{II.4}$$

Where E_s is the energy of $|s\rangle$ and E_{t_i} are the energies of the states $|t_i\rangle$.

Now the interactions with the neighbors $\vec{S}'_1$ and $\vec{S}'_2$ can be added back. Using that J was the strongest bond in the chain, the interactions due to K_1 and K_2 can be treated perturbatively.

$$\tilde{H} = H_0 + \mathcal{H} \tag{II.5}$$

where the perturbation $\mathcal{H}$ is defined as

$$\mathcal{H} = K_1 \vec{S'}_1 \cdot \vec{S}_1 + K_2 \vec{S}_2 \cdot \vec{S'}_2 \tag{II.6}$$

Using second order perturbation theory, the perturbation changes the energy of the ground state as

$$E_s \to E_s + \langle s|\,\mathcal{H}\,|s\rangle + \sum_i |\langle s|\,\mathcal{H}\,|t_i\rangle|^2 \frac{1}{E_s - E_t} \tag{II.7}$$

Carefully computing the matrix elements of $\mathcal{H}$ in the basis of eigenstates of H_0 momentarily regarding $\vec{S'}_1$ and $\vec{S'}_2$ as fixed vectors and using (II.2) leads to $\langle s|\,\mathcal{H}\,|s\rangle = \langle s|\,\mathcal{H}\,|t_0\rangle = 0$ and

$$\begin{aligned} |\langle s|\,\mathcal{H}\,|t_1\rangle|^2 = \frac{1}{8}\big[&K_1^2 S'^{+}_1 S'^{-}_1 + K_2^2 S'^{+}_2 S'^{-}_2 \\ &- K_1 K_2 (S'^{+}_1 S'^{-}_2 + S'^{+}_2 S'^{-}_1)\big] \end{aligned} \tag{II.8}$$

$$\begin{aligned} |\langle s|\,\mathcal{H}\,|t_{-1}\rangle|^2 = \frac{1}{8}\big[&K_2^2 S'^{-}_1 S'^{+}_1 + K_2^2 S'^{-}_2 S'^{+}_2 \\ &- K_1 K_2 (S'^{-}_1 S'^{+}_2 + S'^{-}_2 S'^{+}_1)\big] \end{aligned} \tag{II.9}$$

Putting all this together and grouping according to the K coefficients leads to

$$\begin{aligned} \sum_i |\langle s|\,\mathcal{H}\,|t_i\rangle|^2 \frac{1}{E_s - E_t} = \frac{-2}{J}\Big[&\frac{K_1^2}{8}(S'^{+}_1 S'^{-}_1 + S'^{-}_1 S'^{+}_1) \\ &+ \frac{K_2^2}{8}(S'^{+}_2 S'^{-}_2 + S'^{-}_2 S'^{+}_2) \\ &- \frac{K_1 K_2}{4}(S'^{+}_1 S'^{-}_2 + S'^{+}_2 S'^{-}_1)\Big] \end{aligned} \tag{II.10}$$

Notice that the terms within round parenthesis with K_1^2 and K_2^2 as coefficients are sums of the form $S'^{+}_i S'^{-}_i + S'^{-}_i S'^{+}_i$ (i = 1,2). In each of them, both operators act on the same spin. Since S^+ annihilates an up spin and S^- annihilates a down spin, when acting with this combination of operators on a state of the form $|\uparrow\rangle$ or $|\downarrow\rangle$ (or any superposition of them), one of the terms is always going to be zero and the other will leave the state unchanged. Taking this in account, we can write

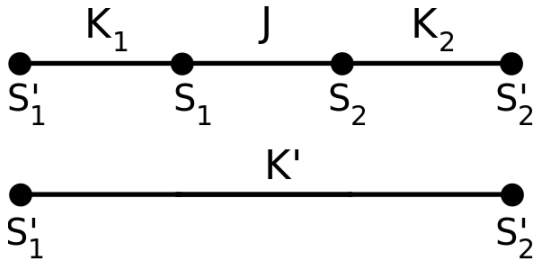

FIG. 1. Spins and couplings involved in one decimation. Top chain depicts the subsystem before the decimation and bottom depicts it after it.

$$\sum_i |\langle s|\,\mathcal{H}\,|t_i\rangle|^2 \frac{1}{E_s - E_t} = \frac{-2}{J}\Big[\frac{K_1^2}{8} + \frac{K_2^2}{8} - \frac{K_1 K_2}{2}(\vec{S'}_1 \cdot \vec{S'}_2)\Big] \tag{II.11}$$

Where in the last term, S_i^+ and S_i^- were written in terms of S_i^x and S_i^y inverting (II.2).

Rearranging and plugging all results in (II.7), we obtain the perturbation to the ground state energy

$$E_s \to E'_s + K' \vec{S'}_1 \cdot \vec{S'}_2 \tag{II.12}$$

where

$$E'_s = -\frac{1}{2}J - \frac{1}{4J}(K_1^2 + K_2^2) \tag{II.13}$$

$$K' = \frac{K_1 K_2}{J} \tag{II.14}$$

We proceed by deleting the spins $\vec{S}_1$ and $\vec{S}_2$ from the original Hamiltonian (I.2) and adding the constant energy E' and an effective coupling between $\vec{S'}_1$ and $\vec{S'}_2$ with coupling constant K' (see figure 1). What has been described so far is a single decimation of the renormalization procedure. We are left with a new Hamiltonian with fewer degrees of freedom and reduced energy scale since we removed its largest bond and replaced it with a weaker effective bond. This means the distribution of bonds P(K) has also been modified. The procedure is continued by repeating the decimation on the modified Hamiltonian with modified P(K), now picking the strongest bond in the set of original bonds with J removed and K' added. We can call this modified set of bonds $\{\tilde{K}_r\}$ and the strongest bond $\Omega = \max\{\tilde{K}\}$. We

will consider the strongest bond of the original distribution equal to 1, so that $0 < \Omega \leq 1$.

Result (II.14), which tells us the strength of the effective bond that replaces the strongest bond, is known as the **Dasgupta-Ma rule** for the XX Heisenberg chain. Dasgupta-Ma renormalization can be derived more generally starting from the XYZ Heisenberg chain defined in (I.1). The Dasgupta-Ma rule for the XYZ model is given by $K'^x = \frac{K_1^x K_2^x}{J^y + J^z}$ and similarly for K'^y and K'^z [2]. Notice how this reduces to the case we derived when setting $K_r^z = 0$ and $K_r^x = K_r^y$, which is the XX model.

We are interested in how the bond distribution changes under repeated decimations of this renormalization process. Recognizing that the distribution is dependent on the energy scale (which is accurately described by the strongest bond Ω), we can write the distribution as P(K,Ω). Now, it is convenient to change our variables into logarithmic expressions[2,3], so we define, using that Ω is the strongest bond in the current Hamiltonian

$$\Gamma = -\ln \Omega \tag{II.15}$$

$$\zeta_r = \ln\Big(\frac{\Omega}{\tilde{K}_r}\Big) \tag{II.16}$$

Notice that as more decimations are performed, the strongest bond Ω is *lowered*, so Γ *increases* with the decimations. Also notice that large ζ_r corresponds to small K_r.

With this definitions, Dasgupta-Ma rule (II.14) becomes

$$\zeta' = \zeta_1 + \zeta_2 - \zeta_\Omega = \zeta_1 + \zeta_2 \tag{II.17}$$

Where ζ_Ω comes from the strongest bond Ω, and is seen from the definition (II.16) that it is zero.

Now the distribution of bonds for a given energy scale in these logarithmic variables can be expressed as $\rho(\zeta, \Gamma)$. The probability of a bond ζ at a fixed scale Γ is written as $\rho(\zeta,\Gamma)d\zeta = d\rho(\zeta,\Gamma)$. What we need is an expression for this probability when the scale is changed to $\Gamma + \delta\Gamma$. We first realize that the change in the distribution has two different sources. The first is the change in the definition of ζ_r in (II.16) due to the change in Γ when the strongest bond gets removed and a new strongest bond is defined, which can be expressed as

$$\zeta \to \zeta' = \ln\Big(\frac{\Omega'}{\tilde{K}}\Big) = \ln\Big(\frac{\Omega}{\tilde{K}}\Big) - \Gamma + \Gamma' = \zeta + \delta\Gamma \tag{II.18}$$

Where the primes denote the values after the decimation.

This first contribution can be expressed in terms of the probability as

$$d\rho(\zeta, \Gamma + \delta\Gamma) \overset{(1)}{=} d\rho(\zeta + \delta\Gamma, \Gamma) \tag{II.19}$$

The second contribution comes from the fact that in the process of eliminating the strongest bond, a new bond of some lower strength is added. This is expressed as

$$d\rho(\zeta, \Gamma + \delta\Gamma) \overset{(2)}{=} \rho(0,\Gamma)\delta\Gamma \times \int_0^\infty d\zeta_L \int_0^\infty d\zeta_R \delta\Big(\zeta - (\zeta_L + \zeta_R)\Big)\rho(\zeta_L,\Gamma)\rho(\zeta_R,\Gamma) \tag{II.20}$$

In this expression, we recognize $\rho(0,\Gamma)\delta\Gamma$ as the the probability of a bond with ζ from zero to $\delta\Gamma$. $\zeta = 0$ is the strongest bond, so in the limit where $\delta\Gamma$ is infinitesimally small, this indicates the value of the distribution at the strongest bond in the current energy scale. The integrals are done over all values of ζ_L and ζ_R, which are the left and right possible neighbors to the strongest bond. Notice the Dasgupta-Ma rule (II.17) inside the delta function in the integral. What this does is ensure that for each value of ζ, we are adding to the new distribution the corresponding probability of obtaining an effective bond with that value.

Putting the two contributions together and taking the limit $\delta\Gamma \to 0$ we obtain the integro-differential equation

$$\frac{\partial\rho(\zeta,\Gamma)}{\partial\Gamma} = \frac{\partial\rho}{\partial\zeta} + \rho(0,\Gamma)\int_0^\infty d\zeta_L \int_0^\infty d\zeta_R \delta\Big(\zeta - (\zeta_L + \zeta_R)\Big)\rho(\zeta_L,\Gamma)\rho(\zeta_R,\Gamma) \tag{II.21}$$

This equation is to be interpreted as a renormalization group flow equation.

III. FIXED POINTS OF THE RENORMALIZATION GROUP FLOW EQUATION

As will be shown, the randomness in the present system always leads to a critical point. This is actually the case for many other random spin chain models. In Renormalization terms, this means randomness is a relevant perturbation to pure critical points[4]. Fixed points of a

RG (renormalization group) flow equation are precisely the critical points of the system, given that a fundamental property of a critical point is their independence on scale. Precisely, we are going to look for fixed points of (II.21) under renormalization (increase of Γ). The following procedure follows Fisher's derivation of the fixed points of this equation[2].

Consider equation (II.21) written in a simplified manner as follows

$$\frac{\partial \rho}{\partial \Gamma} = \frac{\partial \rho}{\partial \zeta} + \rho_0 \rho \otimes_\zeta \rho \tag{III.1}$$

Where $\rho_0 = \rho(0, \Gamma)$ and $\rho \otimes_\zeta \rho$ is the convolution seen in (II.21). To look for appropriate scale invariant points it is useful to explicitly rescale the distribution and the variable ζ by some power κ of Γ

$$\eta = \frac{\zeta}{\Gamma^\kappa} \tag{III.2}$$

$$\rho(\zeta, \Gamma) = \frac{Q(\eta, \Gamma)}{\Gamma^\kappa} \tag{III.3}$$

Equation (III.1) becomes

$$\Gamma \frac{\partial Q}{\partial \Gamma} = \kappa \Big[Q + \eta \frac{\partial Q}{\partial \eta} \Big] + \Gamma^{1-\kappa} \Big[\frac{\partial Q}{\partial \eta} + Q_0 Q \otimes_\eta Q \Big] \tag{III.4}$$

Making $\frac{\partial Q}{\partial \Gamma} = 0$, we obtain the equation we wish to solve. The equation is governed by the first term in square parenthesis when $\kappa > 1$. Ignoring the second term leads to an elementary equation with solution C/η, which is unphysical since it diverges as $\eta \to 0$: the strongest bond. When $\kappa < 1$ the second term dominates and the resulting equation can be solved using Laplace transforms and Adomian polynomials[5], but the solution is also unphysical beacuse it oscillates in sign for big η.

This leaves us only with $\kappa = 1$ as a possible scale exponent, which turns equation (III.4) into

$$0 = Q + \eta \frac{\partial Q}{\partial \eta} + \frac{\partial Q}{\partial \eta} + Q_0 Q \otimes_\eta Q \tag{III.5}$$

Due to the convolution in this equation, it is convenient to use Laplace transforms. Performing the Laplace transformation, we find

$$z \frac{\partial \hat{Q}}{\partial z} = z\hat{Q} + Q_0[\hat{Q}^2 - 1] \tag{III.6}$$

with the condition

$$Q_0 = \lim_{\eta \to 0+} \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \hat{Q} e^{z\eta} d \tag{III.7}$$

The convolution lead to a quadratic term in (III.6). In order to solve this we can perform the substitution

$$\hat{Q} = \frac{-z}{u Q_0} \frac{du}{dz} \tag{III.8}$$

which yields the linear second order differential equation

$$\frac{d^2u}{dz^2} + \frac{1 - z}{z} \frac{du}{dz} - \frac{Q_0^2}{z^2} u = 0 \tag{III.9}$$

which is of a form that can be studied using Frobenius method

Using the form

$$u = \sum_{k=0} A_k z^{k+r} \tag{III.10}$$

and substituting in (III.9), we find

$$\big[r(r-1) + r - Q_0^2\big] A_0 z^{r-2} + \sum_{k=1}^{\infty} \big[(k+r)(k+r-1) + (k+r) - Q_0^2) A_k - (k+r-1) A_{k-1}\big] z^{k+r-2} = 0 \tag{III.11}$$

This is satisfied if the coefficients to all powers of z vanish. Requiring $A_0 \neq 0$ (opposite case leads to a trivial solution), the first term yields the condition $r = \pm Q_0$. Using this and requiring that all coefficients in the sum of the second term also vanish gives us the recurrence relations

$$A_{k+1}^{\pm} = \frac{k \pm Q_0}{(k+1)(k+1 \pm 2Q_0)} A_k^{\pm} \tag{III.12}$$

The solution to (III.9) according to the method of Frobenius depends on the difference between the two values of r found: if $2Q_0 \neq$ integer, the solution is given by

$$u = z^{-Q_0} \sum_{k=0}^{\infty} A_k^- z^k + z^{Q_0} \sum_{k=0}^{\infty} A_k^+ z^k \tag{III.13}$$

On the other hand, if $2Q_0 =$ integer, the solution is given by

$$u = \Big(z^{Q_0}\sum_{k=0}^{\infty} A_k^+ z^k\Big)\Big(1 + C\ln(z)\Big) + z^{-Q_0}\sum_{k=0}^{\infty} D_k z^k \tag{III.14}$$

Where C and D_k are to be determined.

Recapitulating, we have found that according to (III.13) and (III.14), $Q_0 = Q(0)$ parametrizes a family of solutions to equation (III.9), which are to be plugged back into (III.6) to obtain the Laplace transform of the distribution. The inverse Laplace transform of this result is thus the fixed points we were looking for. Fisher[2] determined that for all $Q_0 \neq 1$, the obtained distribution $Q(\eta)$ decays as $1/\eta^{2Q_0+1}$.

Looking back at the original RG flow equation (II.21), we can analyze the behavior of ρ for large ζ. Suppose ρ decays slower than exponentially in the large ζ regime. If this is the case, the convolution $\rho \otimes_\zeta \rho$ is dominated by the regions in its integration domain where ζ_L is small and so ζ_R is large and vice versa. This means that for large ζ we can write

$$\begin{aligned}\rho \otimes_\zeta \rho &\sim 2\int_{large} d\zeta_L \int_{small} d\zeta_R \delta\Big(\zeta - (\zeta_L + \zeta_R)\Big)\rho(\zeta_L)\rho(\zeta_R) \\ &\sim 2\rho(0)\rho(\zeta)\end{aligned} \tag{III.15}$$

Which means that the convolution decays the same as ρ. Thus, the slow decay of ρ makes $\frac{\partial\rho}{\partial\zeta}$ negligible in equation (II.21), which means the form of a subexponentially decaying tail can not be changed by the renormalization: a slowly decaying renormalized distribution must start from an originally slowly decaying distribution. Fisher[2] states that this leads to distributions which are singular at small bond strengths and are not to be discussed further in this work. We turn then to the special case where $Q_0 = 1$.

When $Q_0 = 1$, $2Q_0$ is integer and we have to use the solution (III.14). Considering that it is possible that C = 0 in this expression and that it would yield a solution with the form (III.13), we first check if this is the case. We first check the first term of (III.13) by computing the A_k^-. We see that for $Q_0 = 1$, the series terminates since $A_k^- = 0$ for $k \geq 2$. We find

$$u_1 = z^{-Q_0}\sum_{k=0}^{\infty} A_k^- z^k = \frac{1}{z} + 1 \tag{III.16}$$

Taking two derivatives and substituting in (III.9) shows u_1 is in fact a solution. For the second term of (III.13) we obtain using the recursion relation for A_k^+ that

$$u_2 = z^{Q_0}\sum_{k=0}^{\infty} A_k^+ z^k = 2\big(-\frac{1}{z} - 1 + \frac{e^z}{z}\big) \tag{III.17}$$

which also proves to be a solution to (III.9). This proves that in fact C = 0 in (III.13) and there is no logarithmic term in the solution. We can then write the general solution as

$$u = \frac{\alpha(1+z) + \beta(1+z-e^z)}{z} \tag{III.18}$$

We can now compute $\hat{Q}$ using (III.8). This yields

$$\hat{Q} = \frac{\alpha' + \beta e^z(1-z)}{\alpha'(1+z) + \beta e^z} \tag{III.19}$$

where $\alpha' = \alpha + \beta$. Considering that the distribution $Q(\eta)$ has a cutoff value at $\eta = 0$, the behavior of the Laplace transform at large z should be $\hat{Q}(z) \sim Q_0/z$, which is the Laplace transform of the Heaviside step function. This is only possible if we have $\beta = 0$. Taking $\alpha' = 1$, we have $\hat{Q} = 1/(1+z)$ and so we obtain that the fixed point in the RG flow equation corresponds to the distribution

$$Q^*(\eta) = e^{-\eta}\theta(\eta) \tag{III.20}$$

Where θ is the Heaviside step function. This is known as the **random singlet fixed point distribution**. Fisher[2] proves this is a stable fixed point by linearizing the flow equation and showing that perturbations to this solution decay exponentially. This can be written again in terms of the original distribution by changing back to the variable ζ inverting the transformation (III.2) remembering that we proved $\kappa = 1$

$$\rho^*(\zeta, \Gamma) = \frac{1}{\Gamma} e^{-\zeta/\Gamma}\theta(\zeta) \tag{III.21}$$

We can go further and revert the transformation to logarithmic variables (II.15) and (II.16) to obtain the distribution in terms of the bond strengths K and strongest bond Ω

$$P^*(K, \Omega) = \frac{\alpha}{\Omega}\Big[\frac{\Omega}{K}\Big]^{1-\alpha}\theta(\Omega - K) \tag{III.22}$$

where

$$\alpha = 1/\Gamma = -1/\ln\Omega \tag{III.23}$$

Notice that the rescaling change of variable (III.2) is necessary for the distribution to be a fixed point. Before Fisher[2] found the fixed point (III.20), Dasgupta and Ma[1] had guessed (III.22) when trying to find a universal power-law for the model, using α as the universal exponent. They called the distribution an "almost fixed point", as they found that for small α, equation (III.22) is approximately correct if one accounts for variations of α as a function of Ω. This is exactly what we found by obtaining (III.22) from finding first the fixed point in the rescaled variables, which lead to this form with α defined as in (III.23).

FIG. 2. Schematic of a section of a chain in the random singlet phase.

IV. THE RANDOM SINGLET PHASE

The upshot of the RG procedure we described and the fixed point found is that the Heisenberg XX antiferromagnetic chain with quenched randomness at zero temperature is driven at long distances to a critical point known as the random singlet phase, described by the distribution (III.21)[4]. This is true for other random spin chain models and can be proven similarly by using the appropriate Dasgupta-Ma rule to construct the renormalization flow equation.

The random singlet phase consists on singlets randomly formed over arbitrarily long length scales in a way that bonds never cross (see figure 2). It is easy to imagine this remembering each step of the Dasgupta-Ma renormalization. We paired the two spins with strongest bond in a singlet and replaced them with an effective bond between the neighboring spins. After repeating this removal of spins several times, the total effective bond between two spins that are far away can become the strongest bond of the chain and be then paired in a long ranged singlet.

As a final result, we are going to estimate the typical distance between spins at a given energy scale Γ in this phase. The probability of having spins connected with the strength Ω is $\rho(0,\Gamma)$, so a fraction $2\rho(0,\Gamma)d\Gamma$ of the spins left at scale Γ are removed at this renormalization step. Referring to (III.21), near the fixed point $\rho(0,\Gamma) \approx 1/\Gamma$. We can then write the change in number of spins n due to renormalization as

$$\frac{dn}{d\Gamma} = \frac{-2n}{\Gamma} \tag{IV.1}$$

which means

$$n(\Gamma) = \frac{1}{\Gamma^2} \tag{IV.2}$$

The typical length between the remaining spins at energy scale Γ is then

$$L(\Gamma) \sim \frac{1}{n(\Gamma)} = \Gamma^2 = \left[\ln\left(\frac{1}{\Omega}\right)\right]^2 \tag{IV.3}$$

Many other interesting properties can be derived from this system from the random singlet distribution, such as spin correlations and entanglement entropy.

[1] C. Dasgupta and S. K. Ma, Phys. Rev. B 22, 1305 (1980)
[2] D. S. Fisher, Phys. Rev. B 50, 3799 (1994).
[3] Fagotti, M. *Entanglement Correlations in exactly solvable models*, PhD thesis, University of Pisa (2011)
[4] G. Refael and J. E. Moore, cond-mat/0406737v1 (2004)
[5] G. Adomian and R. Rach, Journal of Mathematical Analysis and Applications 114, 171 (1986)