Boundary Conformal Field Theory

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1 Abstract

Conformal field theories are quantum field theories that are invariant under conformal transformations; they have several important applications, including in condensed matter physics and string theory. Boundary conformal field theories are CFTs on a domain with a boundary, which has consequences for basic quantities in the theory such as Ward identities, the stress-energy tensor, and also the conformal weights.

Our aim is to examine some of the implications of introducing boundary conditions to a CFT. We begin by recounting a few basics of regular CFT; namely, the stress-energy tensor T, operator product expansions (OPEs), Ward Identities, and primary operators. We present a detailed derivation of the Ward Identities, then proceed to consider how these change under boundary conditions. We present a derivation of the BCFT Ward Identity. Lastly, we examine how consideration of CFTs on different, conformally related manifolds can reveal some of the underlying structure of the CFT.

2 Conformal field theory

In this section we will compute the conformal Ward identities in a domain without boundaries.

2.1 The stress-energy tensor

Consider a manifold $\mathcal{M}$, described by a metric $g_{\alpha\beta}$, and a general infinitesimal transformation:

$$\sigma^\mu \to \sigma'^\mu = \sigma^\mu + \epsilon^\mu(\sigma)$$

We can define the stress-energy tensor $T_{\mu\nu}$ in terms of the variation of the action:

$$\delta S = \frac{1}{2\pi} \int \sqrt{g} T^{\mu\nu} \partial_\nu \epsilon_\mu \mathrm{d}^2\sigma \tag{2.1}$$

where $\{\sigma^\mu\}$ are a set of Euclidean coordinates. A few comments can be made about this expression:

1. if the action S is invariant under translations and rotations, then $T_{\mu\nu}$ is a conserved and symmetric quantity
2. if the action S is scale invariant, then $T_{\mu\nu}$ is traceless

Now we switch to complex coordinates $\{z, \bar{z}\}$. Imposing the conditions mentioned above, along with the conservation law, the components of the stress-energy tensor are:

$$T_{z\bar{z}} = T_{\bar{z}z} = 0$$
$$\partial^{\bar{z}} T_{zz} = \partial^{z} T_{\bar{z}\bar{z}} = 0$$

The last expression, in particular, tells us that the components T_{zz} and $T_{\bar{z}\bar{z}}$ are respectively holomorphic and anti-holomorphic. We can redefine them as:

$$T_{zz} = T(z)$$

$$\bar{T}_{\bar{z}\bar{z}} = \bar{T}(\bar{z})$$

This is a key feature of our calculations, since it allows us to treat z and $\bar{z}$ as independent variables and therefore to treat separately the holomorphic and the anti-holomorphic parts of the theory.
We can also define the conserved currents J and $\bar{J}$ as:

$$\begin{cases} J = T(z)\epsilon(z) \\ \bar{J} = \bar{T}(\bar{z})\bar{\epsilon}(\bar{z}) \end{cases} \tag{2.2}$$

2.2 Operator Product Expansion

Consider a CFT which contains some set of local operators $\{\mathcal{O}_i\}$. Then an operator product expansion (OPE) approximates two local operators at nearby points inside time-ordered correlation functions by a string of operators at one of these points:

$$\langle \mathcal{O}_i(z,\bar{z})\mathcal{O}_j(w,\bar{w})\dots\rangle = \sum_k \mathcal{C}_{ij}^k(z-w,\bar{z}-\bar{w})\langle \mathcal{O}_k(w,\bar{w})\dots\rangle \tag{2.3}$$

where the $\dots$ represents some arbitrary set of other operator insertions, as long as these other insertions are at a sufficiently large distance compared to $|z-w|$. The OPE is then, in fact, an exact statement with a radius of convergence equal to the distance to the next nearest insertion. In our derivations we will often omit them, but the expressions we write must always be intended as inside the brakets.

2.3 Ward identity

To derive the Ward identities we will use the path integral formalism. We will at first derive our out theory changes after an infinitesimal transformation, then we will proceed imposing that transformation to be conformal.
Consider the following path integral:

$$Z = \int e^{-S[\phi]}\mathcal{D}\phi \tag{2.4}$$

we want to see how it changes if we make an infinitesimal transformation of the form $\phi \to \phi' = \phi + \epsilon(\sigma)\delta\phi$.

$$Z \to Z' = \int e^{-S[\phi']}\mathcal{D}\phi' = \int e^{-S[\phi]-\delta S}\mathcal{D}\phi' = \int e^{-S[\phi]}e^{-\delta S}\mathcal{D}\phi'$$

Now we can expand the term $e^{-\delta S} \simeq 1 - \delta S$ and rewrite the expression as:

$$\int e^{-S[\phi]}\left(1 - \frac{1}{2\pi}\int \mathrm{d}^2\sigma\sqrt{g}J^\alpha\partial_\alpha\epsilon(\sigma)\right)\mathcal{D}\phi' \tag{2.5}$$

We want the infinitesimal transformation to be a symmetry of our theory, and this requires that it leaves invariant both the action and the measure of the path integral:

1. $S[\phi'] = S[\phi]$
2. $\mathcal{D}\phi' = \mathcal{D}\phi$

These conditions together imply that $\int \mathcal{D}\phi' e^{-S[\phi']} = \int \mathcal{D}\phi e^{-S[\phi]}$ and therefore:

$$\frac{1}{2\pi}\int \mathcal{D}\phi e^{-S[\phi]}\left(\int \mathrm{d}^2\sigma\sqrt{g}J^\alpha\partial_\alpha\epsilon(\sigma)\right) = 0 \tag{2.6}$$

Using integration by parts, we can rewrite the term $J^\alpha \partial_\alpha \epsilon = \partial_\alpha (J^\alpha \epsilon) - \partial_\alpha J^\alpha \epsilon$ and since the total derivative term can be neglected, we find ourselves with

$$\frac{1}{2\pi} \int \mathcal{D}\phi e^{-S[\phi]} \Big(\int \mathrm{d}^2\sigma \sqrt{g} \partial_\alpha J^\alpha \epsilon \Big) = 0 \tag{2.7}$$

This equation must hold for any ϵ therefore:

$$\partial_\alpha J^\alpha = 0 \tag{2.8}$$

which is Noether's theorem.
Consider now the correlation function between N operators $\mathcal{O}_i$:

$$\langle \mathcal{O}_1 \dots \mathcal{O}_N \rangle = \frac{1}{\mathcal{Z}} \int \mathcal{D}_\phi e^{-S[\phi]} \mathcal{O}_1 \dots \mathcal{O}_N \tag{2.9}$$

We want to consider a conformal transformation. However, instead of having it acting on the entire manifold $\mathcal{M}$ we will define it as being non-zero only in a region around an operator, for example $\mathcal{O}_1$ acting on the point σ_1. This is equivalent to choosing a contour $\mathcal{C}$ around σ_1 such that the operator will transform as:

$$\mathcal{O}_1 \to \mathcal{O}_1' = \mathcal{O}_1 + \epsilon(\sigma)\delta\mathcal{O}_1 \tag{2.10}$$

with $\epsilon = 0$ outside the contour $\mathcal{C}$.

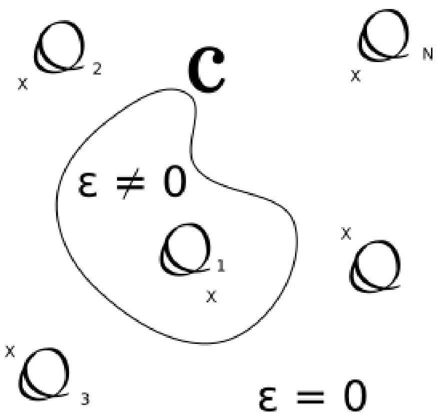


Figure 1: Definition of the contour $\mathcal{C}$

Using equation (2.5), we can rewrite (2.9) as

$$\langle \mathcal{O}_1' \dots \mathcal{O}_N' \rangle = \frac{1}{Z} \int \mathcal{D}\phi e^{-S[\phi]} \Big(1 + \frac{1}{2\pi} \int_{\mathcal{C}} d^2\sigma \sqrt{g} \partial_\alpha J^\alpha \epsilon \Big) (\mathcal{O}_1 + \epsilon\delta\mathcal{O}_1) \mathcal{O}_2 \dots \mathcal{O}_N = \tag{2.11}$$

$$\frac{1}{Z} \int \mathcal{D}\phi e^{-S[\phi]} \mathcal{O}_1 \dots \mathcal{O}_N + \frac{1}{2\pi} \frac{1}{Z} \int \mathcal{D}\phi e^{-S} \int_{\mathcal{C}} \mathrm{d}^2\sigma \sqrt{g} \partial_\alpha J^\alpha \epsilon \mathcal{O}_1 \dots \mathcal{O}_N + \frac{1}{Z} \int \mathcal{D}\phi e^{-S} \epsilon \delta\mathcal{O}_1 \dots \mathcal{O}_N + o(\epsilon^2)$$

$$= \langle \mathcal{O}_1 \dots \mathcal{O}_N \rangle + \frac{1}{2\pi} \int_{\mathcal{C}} \mathrm{d}^2\sigma \sqrt{g} \partial_\alpha \langle J^\alpha \mathcal{O}_1 \dots \mathcal{O}_N \rangle + \langle \delta \mathcal{O}_1 \dots \mathcal{O}_N \rangle = \langle \mathcal{O}_1 \dots \mathcal{O}_N \rangle$$

Once again if we want to consider a transformation that is a symmetry we need this equation to hold for any ϵ, which means:

$$\langle \delta \mathcal{O}_1 \dots \mathcal{O}_N \rangle = -\frac{1}{2\pi} \int_{\mathcal{C}} \mathrm{d}^2\sigma \sqrt{g} \partial_\alpha \langle J^\alpha \mathcal{O}_1 \dots \mathcal{O}_N \rangle \tag{2.12}$$

This is the Ward identity for a generic symmetry.
We can easily apply this result to the specific case of a conformal symmetry with a few more steps. First we are going to use the Stokes' theorem:

$$\int_{\mathcal{C}} \mathrm{d}^2\sigma \sqrt{g} \partial_\alpha J^\alpha = \oint_{\partial\mathcal{C}} J^\alpha \hat{n}_\alpha \sqrt{g} \mathrm{d}^2\sigma = \oint_{\partial\mathcal{C}} (J_1 \mathrm{d}\sigma_2 - J_2 \mathrm{d}\sigma_1) = -i \oint_{\partial\mathcal{C}} (J_z \mathrm{d}z - J_{\bar{z}} \mathrm{d}\bar{z})$$

Where in the last step we switched to complex coordinates $z, \bar{z}$. The Ward identity then becomes:

$$\langle \delta \mathcal{O}_1 \dots \rangle = \frac{i}{2\pi} \oint_{\partial\mathcal{C}} \mathrm{d}z \langle J_z(z,\bar{z}) \mathcal{O}_1 \dots \rangle - \frac{i}{2\pi} \oint_{\partial\mathcal{C}} \mathrm{d}\bar{z} \langle J_{\bar{z}}(z,\bar{z}) \mathcal{O}_1 \dots \rangle \tag{2.13}$$

where now the coordinates $\{\sigma_i\}$ have become complex too, $\{w, \bar{w}\}$. At this point we use that fact that, for a conformal symmetry, the conserved currents J and $\bar{J}$ are, respectively, holomorphic and anti-holomorphic. Therefore we can apply to those integrals Cauchy's theorem and, treating once again z and $\bar{z}$ as independent variables, the Ward identities can be split in the following two equations:

$$\delta \mathcal{O}_1(w,\bar{w}) = \frac{i}{2\pi} \oint_{\partial\mathcal{C}} \mathrm{d}z J(z) \mathcal{O}_1(w,\bar{w}) = -Res[J(z)\mathcal{O}_1(w,\bar{w})] \tag{2.14}$$

$$\delta \mathcal{O}_1(w,\bar{w}) = -\frac{i}{2\pi} \oint_{\partial\mathcal{C}} \mathrm{d}z \bar{J}(\bar{z}) \mathcal{O}_1(w,\bar{w}) = -Res[\bar{J}(\bar{z})\mathcal{O}_1(w,\bar{w})] \tag{2.15}$$

Note: the boundary $\partial\mathcal{C}$ is taken, in the second integral, in opposite direction with respect to the first integral.
We can rewrite equations (2.14) and (2.15) using equation (2.2)

$$\delta \mathcal{O}_1(w,\bar{w}) = -Res[\epsilon(z) T(z) \mathcal{O}_1(w,\bar{w})] \tag{2.16}$$

$$\delta \mathcal{O}_1(w,\bar{w}) = -Res[\bar{\epsilon}(\bar{z}) \bar{T}(\bar{z}) \mathcal{O}_1(w,\bar{w})]$$

2.4 Primary operators

We are mainly interested in primary operators, since all the others are descendants, and knowing the primaries would allow us to know all the coefficients in the OPE, and therefore the whole theory. Under a conformal transformation of the form

$$\begin{cases} z \to w = z + \epsilon(z) \\ \bar{z} \to \bar{w} = \bar{z} + \bar{\epsilon}(\bar{z}) \end{cases}$$

a primary operator transforms in the following way:

$$\mathcal{O}(w,\bar{w}) = \left(\frac{\partial w}{\partial z}\right)^h \left(\frac{\partial \bar{w}}{\partial \bar{z}}\right)^{\bar{h}} \mathcal{O}(z,\bar{z}) \tag{2.17}$$

where h and $\bar{h}$ are the conformal weights.
Once again we will consider the coordinates z and $\bar{z}$ to be independent and we will treat separately the holomorphic and the anti-holomorpic parts. For an infinitesimal transformation, $\epsilon(z)$ is small enough to allow us to expand the derivatives:

$$\left(\frac{\partial w}{\partial z}\right)^h = (1+\partial_z\epsilon(z))^h \simeq (1+h\partial_z\epsilon(z))$$

Using the Taylor expansion we can also expand the operator $\mathcal{O}(z,\bar{z})$ around $\{w,\bar{w}\}$

$$\mathcal{O}(z) \simeq \mathcal{O}(w) + \partial_z\mathcal{O}(w)(z-w) + o(z^2) = \mathcal{O}(w) - \epsilon(z)\partial_z\mathcal{O}(w) + o(z^2)$$

We can derive the variation of an operator under conformal transformation as follows:

$$\mathcal{O}(w) = \left(\frac{\partial w}{\partial z}\right)^h \mathcal{O}(z) \simeq (1+h\partial_z\epsilon(z))(1-\epsilon(z)\partial_z)\mathcal{O}(w)$$

$$= \mathcal{O}(w) + h\partial_z\epsilon(z)\mathcal{O}(w) - \epsilon(z)\partial_z\mathcal{O}(w) + o(\epsilon^2)$$

Therefore the variation of a primary operator under a conformal transformation is

$$\delta\mathcal{O}(w) = (h\partial_z\epsilon(z) - \epsilon(z)\partial_z)\mathcal{O}(w) \tag{2.18}$$

We can repeat the same calculations for the anti-holomorphic part, to get:

$$\delta\mathcal{O}(\bar{w}) = (\bar{h}\partial_{\bar{z}}\bar{\epsilon}(\bar{z}) - \bar{\epsilon}(\bar{z})\partial_{\bar{z}})\mathcal{O}(\bar{w}) \tag{2.19}$$

We can now equate the equations (2.16) with (2.18)

$$-Res[\epsilon(z)T(z)\mathcal{O}(w,\bar{w})] = (h\partial_z\epsilon(z) - \epsilon(z)\partial_z)\mathcal{O}(w,\bar{w})$$

$$\epsilon(z)T(z)\mathcal{O}(w,\bar{w}) = \frac{-h\partial_z\epsilon(z) + \epsilon(z)\partial_z}{z-w}\mathcal{O}(w,\bar{w})$$

We can now get rid of $\epsilon(z)$ recalling that $\epsilon(z) = w - z$, therefore $\partial_z\epsilon(z) = -1$ which leads to:

$$T(z)\mathcal{O}(w,\bar{w}) = \frac{h\mathcal{O}(w,\bar{w})}{(z-w)^2} + \frac{\partial_z\mathcal{O}(w,\bar{w})}{(z-w)} + \dots \tag{2.20}$$

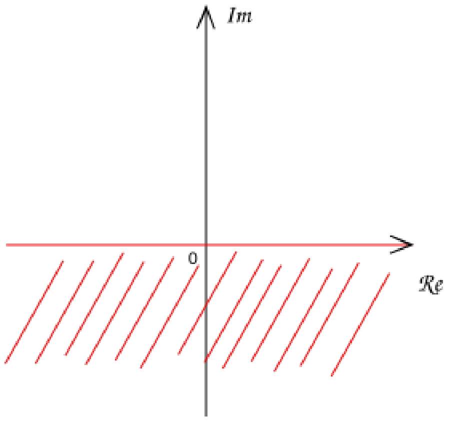
and for the anti-holomorphic part as well:

$$\bar{T}(\bar{z})\mathcal{O}(w,\bar{w}) = \frac{\bar{h}\mathcal{O}(w,\bar{w})}{(\bar{z}-\bar{w})^2} + \frac{\partial_{\bar{z}}\mathcal{O}(w,\bar{w})}{(\bar{z}-\bar{w})} + \dots \tag{2.21}$$

In equations (2.20) and (2.21) we indicated with "..." the non singular terms of the OPE. We found the Ward identities for a conformal transformation.

3 Boundary CFT

We want to see how the theory changes if we impose a boundary on our manifold. So far we have been working on a flat, 2D manifold, and we derived the Ward identities switching to complex coordinates. In particular, we will take the boundary to be the real axis, and consider only the upper half of the complex plane.



A quick comment can be made here about the stress energy tensor. Since our manifold is flat, we found in section 2 that the off-diagonal components of the stress-energy tensor are zero. However, this is not the case if we have a curved manifold. In that case, an immediate consequence of imposing the boundary, would be to require the off-diagonal components of T to vanish on the boundary. These components in fact, represent the flow of momentum and energy, therefore imposing that they are zero is equivalent to imposing the conservation laws. This is called the **conformal boundary condition**.

As a result of our choice of boundary, the operators $\mathcal{O}(z, \bar{z})$ are defined only on the upper half of the complex plane. Moreover the stress-energy tensor will no longer split into a holomorphic and an anti-holomorphic part anymore.
The way in which we are going to attack this problem is somehow akin to the "method of images" technique in electrodynamics: the potential for certain complicated charge distributions can be found considering a simpler distribution that satisfies the same boundary conditions. What we will do is to let the operator $\mathcal{O}_1$ vary over the whole complex field, introducing a redefined stress-energy tensor T in the following way:

$$T(z, \bar{z}) = \begin{cases} T(z) & \Im m(z) \geq 0 \\ \bar{T}(\bar{z}) & \Im m(z) < 0 \end{cases} \tag{3.1}$$

so that it satisfies the boundary condition, since on the real axis $z = \bar{z}$. At this point however we can no longer treat the the variables z and $\bar{z}$ independently. This means that we cannot separate the holomorphic and the anti-holomorphic parts of the theory as we did in the previous section.

Consider the same conformal transformation as before:

$$\begin{cases} z \to w = z + \epsilon(z) \\ \bar{z} \to \bar{w} = \bar{w} + \bar{\epsilon}(\bar{z}) \end{cases}$$

If we want to preserve the geometry of our boundary, it is necessary that the transformation is such that $\epsilon(\bar{z}) = \bar{\epsilon}(\bar{z})$. The first part of the previous derivation still holds, until equation (2.12)

$$\langle \delta\mathcal{O}_1 \dots \mathcal{O}_N \rangle = -\frac{1}{2\pi} \int_{\mathcal{C}} \mathrm{d}^2\sigma \sqrt{g} \partial_\alpha \langle J^\alpha \mathcal{O}_1 \dots \mathcal{O}_N \rangle \tag{3.2}$$

Once again we will use the Stokes' theorem and switching to complex coordinates we will obtain

$$\langle \delta\mathcal{O}_1 \dots \rangle = \frac{i}{2\pi} \oint \mathrm{d}z \langle \epsilon(z) T(z,\bar{z}) \mathcal{O}_1 \dots \rangle = \frac{i}{2\pi} \oint_{\partial\mathcal{C}} \mathrm{d}z \langle \epsilon(z) T(z) \mathcal{O}_1 \dots \rangle + \frac{i}{2\pi} \oint_{\partial\bar{\mathcal{C}}} \mathrm{d}z \langle \epsilon(z) \bar{T}(\bar{z}) \mathcal{O}_1 \dots \rangle$$

where $\partial\bar{\mathcal{C}}$ is a contour symmetric to $\partial\mathcal{C}$ that lies in the lower half of the plane.

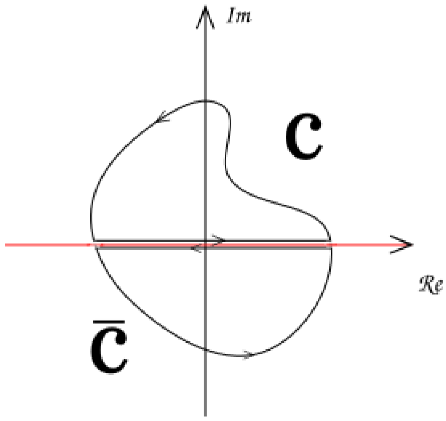


Figure 2: The integrals over the two straight portions of $\partial\mathcal{C}$ and $\partial\bar{\mathcal{C}}$ cancel out

Note: in section 2 we found ourselves with two complex integral due to the fact that we were considering separately $T(z)$ and $\bar{T}(\bar{z})$, while this time the two integrals come from the fact that we defined $T(z,\bar{z})$ by parts.
Once again, using Cauchy's theorem we can write

$$\delta\mathcal{O} = -Res[\epsilon(z)T(z)\mathcal{O}] - Res[\epsilon(z)\bar{T}(z)\mathcal{O}] \tag{3.3}$$

The variation of the operator can be found similarly as we did for equation (2.18):

$$\delta\mathcal{O}_1(w,\bar{w}) = \left(\frac{\partial w}{\partial z}\right)^h \left(\frac{\partial \bar{w}}{\partial \bar{z}}\right)^{\bar{h}} \mathcal{O}_1(z,\bar{z}) =$$

$$(1 + h\partial_z\epsilon)(1 + \bar{h}\partial_{\bar{z}}\epsilon)(\mathcal{O}_1(w,\bar{w}) + \partial_z\mathcal{O}_1(w,\bar{w})(z-w) + \partial_{\bar{z}}\mathcal{O}_1(w,\bar{w})(\bar{z}-\bar{w})) + o(z^2,\bar{z}^2)$$

$$= \mathcal{O}_1(w,\bar{w}) + h\partial_z\epsilon\mathcal{O}_1(w,\bar{w}) + \bar{h}\partial_{\bar{z}}\epsilon\mathcal{O}_1(w,\bar{w}) + \partial_z\mathcal{O}_1(w,\bar{w}) + \partial_{\bar{z}}\mathcal{O}_1(w,\bar{w}) + o(z^2,\bar{z}^2)$$

The last step is to compare this result with equation (3.3), to get the conformal Ward identity:

$$T(z)\mathcal{O}(w,\bar{w}) = \frac{h\mathcal{O}(w,\bar{w})}{(z-w)^2} + \frac{\partial_z\mathcal{O}(w,\bar{w})}{(z-w)} + \frac{\bar{h}\mathcal{O}(w,\bar{w})}{(\bar{z}-\bar{w})^2} + \frac{\partial_{\bar{z}}\mathcal{O}(w,\bar{w})}{(\bar{z}-\bar{w})} \tag{3.4}$$

This is the Ward identity for a BCFT with boundary on the real axis.

4 The Annulus Partition Function

4.1 Virasoro Algebra

A useful mathematical structure for examining CFTs is the Virasoro Algebra. Because of scale invariance, one may quantise states on a circle, in which case the Hamiltonian is the dilatation operator $\hat{D}$, which is defined as

$$\hat{D} = \frac{1}{2\pi}\int_0^{2\pi} r\hat{T}_{rr} r d\theta = \frac{1}{2\pi i}\int_C z\hat{T}(z)dz - \frac{1}{2\pi i}\int_C \bar{z}\hat{\bar{T}}(\bar{z})d\bar{z} \tag{4.1}$$

C can be any contour that contains the origin, due to Cauchy's theorem. It is then convenient to define the operators $\hat{L}_n$:

$$\hat{L}_n \equiv (1/2\pi)\int_C z^{n+1}\hat{T}(z)dz \tag{4.2}$$

and

$$\hat{\bar{L}}_n \equiv (1/2\pi)\int_C \bar{z}^{n+1}\hat{\bar{T}}(\bar{z})d\bar{z} \tag{4.3}$$

so that we can rewrite (4.1) as

$$\hat{D} \equiv \hat{L}_0 + \hat{\bar{L}}_0 \tag{4.4}$$

Using the TT-OPE, the OPE of $\hat{T}$ with itself, one can work out the Virasoro Algebra $\mathcal{V}$:

$$[\hat{L}_n, \hat{L}_m] = (n-m)\hat{L}_{n+m} + \frac{c}{12}n(n^2-1)\delta_{n+m,0} \tag{4.5}$$

and similarly for $\bar{\mathcal{V}}$:

$$[\hat{\bar{L}}_n, \hat{\bar{L}}_m] = (n-m)\hat{\bar{L}}_{n+m} + \frac{c}{12}n(n^2-1)\delta_{n+m,0}. \tag{4.6}$$

The state $|\phi_j\rangle \equiv \hat{\phi}_j(0,0)|0\rangle$, where $|0\rangle$ is the vacuum state and $\hat{\phi}_j$ is the scaling operator, is an eigenstate of both the $\hat{L}_0$ and $\hat{\bar{L}}_0$ operators:

$$\hat{L}_0|\phi_j\rangle = h_j|\phi_j\rangle \tag{4.7}$$

$$\hat{\bar{L}}_0|\phi_j\rangle = \bar{h}_j|\phi_j\rangle \tag{4.8}$$

The full Hilbert space of the CFT is

$$\mathcal{H} = \bigoplus_{h,\bar{h}} n_{h,\bar{h}}\mathcal{V}_h \otimes \bar{\mathcal{V}}_{\bar{h}}. \tag{4.9}$$

4.2 CFT on a cylinder

Consider the conformal transformation

$$z \to z' = e^{-iz} \tag{4.10}$$

where

$$z = \sigma + i\tau \tag{4.11}$$

with $\sigma \in [0, 2\pi)$ is a parameterization of a Euclidean cylinder. This transformation maps a Euclidean cylinder to the complex plane. Since CFTs are invariant under conformal transformations, studying a particular CFT on either of these manifolds is essentially the same as studying it on the other; anything that holds for that CFT on either of these manifolds automatically applies to the same theory on the other.

Under conformal transformations, the stress-energy tensor T transforms as

$$T'(z') = \left(\frac{\partial z'}{\partial z}\right)^{-2} \left[T(z) - \frac{c}{12} S(z', z)\right] \tag{4.12}$$

where the Schwarzian $S(z', z)$ is given by

$$S(z', z) = \left(\frac{\partial^3 z'}{\partial z^3}\right)\left(\frac{\partial z'}{\partial z}\right)^{-1} - \frac{3}{2}\left(\frac{\partial^2 z'}{\partial z^2}\right)^2\left(\frac{\partial z'}{\partial z}\right)^{-2} \tag{4.13}$$

Then, from the Hamiltonian

$$H = \int d\sigma T_{\tau\tau} = -\int d\sigma (T_{ww} + \bar{T}_{\bar{w}\bar{w}}) \tag{4.14}$$

it is possible to determine that the ground state energy on the cylinder is

$$E = -\frac{2\pi(c + \tilde{c})}{24}, \tag{4.15}$$

where c and $\tilde{c}$ are the left- and right- moving central charges, respectively. This is the negative Casimir energy on a cylinder.

4.3 Boundary CFT on an annulus

Something similar can be done for a conformal field theory with a boundary by considering a CFT on an annulus. The annulus can be seen as a rectangle of unit width and height δ, with two edges identified. The remaining two edges have boundary conditions a and b on them.

The partition function on the annulus, $Z_{ab}(\delta)$, is given by

$$Z_{ab}(\delta) = \text{Tr} e^{-\delta \hat{H}_{ab}} \tag{4.16}$$

The Hamiltonian $\hat{H}$ is the generator for infinitesimal translations along an infinitely long strip of unit width,

$$\hat{H}_{ab} = \pi\hat{D} - \pi c/24 = \hat{L}_0 - \pi c/24. \tag{4.17}$$

where the last equality follows from the fact that $L_n = \bar{L}_{-n}$. This is a consequence of the stress-energy tensor's transformation law (4.12) applied to the circle as well as the aforementioned conformal boundary condition. Thus, the partition function can be written as

$$Z_{ab}(\delta) = Tr e^{-\pi\delta(\hat{L}_0 - c/24)} = \sum_h n_h^{ab} \chi_h(q) \tag{4.18}$$

where $q \equiv e^{-\pi\delta}$ and in the last expression the partition function has been decomposed into characters.

Next consider this same partition function, this time as the path integral of a CFT propagated for an imaginary time δ^{-1} on a unit circle:

$$Z_{ab}(\delta) = \langle a|e^{-\hat{H}/\delta}|b\rangle \tag{4.19}$$

where the Hamiltonian is now $\hat{H} = 2\pi(\hat{L}_0 + \hat{\bar{L}}_0) - \frac{\pi c}{6}$. The states $|a\rangle$ and $|b\rangle$ are boundary states that belong to $\mathcal{H}$ (4.9).

Again using the fact that $L_n = \bar{L}_{-n}$, any boundary state $|B\rangle$ must satisfy

$$\hat{L}_n|B\rangle = \hat{\bar{L}}_{-n}|B\rangle; \tag{4.20}$$

these states form a subspace of $\mathcal{H}$. Furthermore, because of the decomposition (4.9), $|B\rangle$ can also be written as a linear combination of states from $\mathcal{V}_h \otimes \bar{\mathcal{V}}_{\bar{h}}$. Taking $n = 0$ in (4.20) and using the eigenvalue equations (4.7) then implies $h = \bar{h}$.

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