Constructing integrable vertex models with gauge theory

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Abstract

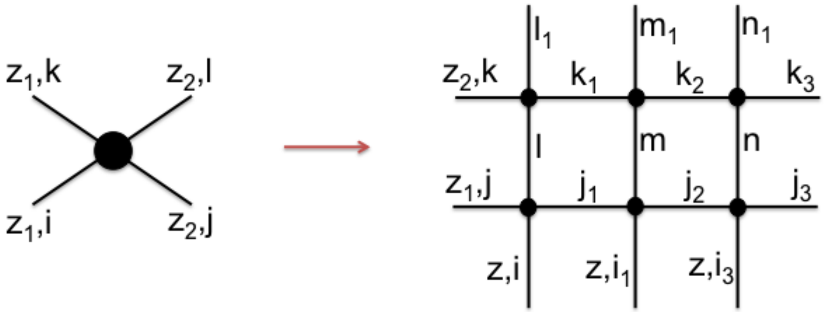
In this digest we will give the reader a lightning tour of a remarkable correspondence between integrable vertex models and gauge theory, which was recently discovered in a rather abstract guise in [1, 2] and expanded upon in [3, 4], where it was furthermore introduced in a more common setting and language. In short, the basis of this correspondence is the fact that some solutions of a particular type of 4D gauge theory, when projected upon the space(time) in which the 2D or $(1 + 1)$D integrable system lives, solve the "classical" Yang-Baxter equation of integrable physics. These solutions moreover display how the rich algebraic structure that accompanies integrability manifests itself in the language of gauge theory. Due to the nature of this document quite a lot of material and detail will be omitted; the reader is encouraged to consult the references.

I. Integrability for vertex models

Some classical statistical systems in 2D space and quantum many-body systems in $(1+1)$D spacetime enjoy a description in terms of vertices spread across some 2D topological manifold and interconnected by curves [5–8]. In one of the simplest cases, we find that such a picture emerges for a system of contact-interacting particles in 1D space, which is fully characterized by the crossing of particle worldlines in the corresponding $(1+1)$D spacetime. In particular, these crossings signify the particle collisions and are the sole points of interest as this theory only contains a contact interaction. In other words, the states in such a theory are fully characterized by what occurs at the vertices of a 2D network of worldlines.

Because these vertices, being simply the nodes in a graph, characterize the topology of our vertex-network, we find that the states in a vertex model are in fact labeled by this topology and are hence diffeomorphism invariant. Therefore in defining such a model we merely require the underlying base manifold to have topological structure. Therefore in defining such a model we merely require the underlying base manifold to have topological structure. This topological vertex situation should be compared with e.g. an Ising model, where spins on *different* sites interact and which consequently requires a notion of distance, as to distinguish nearest- from next-nearest neighbors etc. In other words, we would need the more complicated metric manifold.

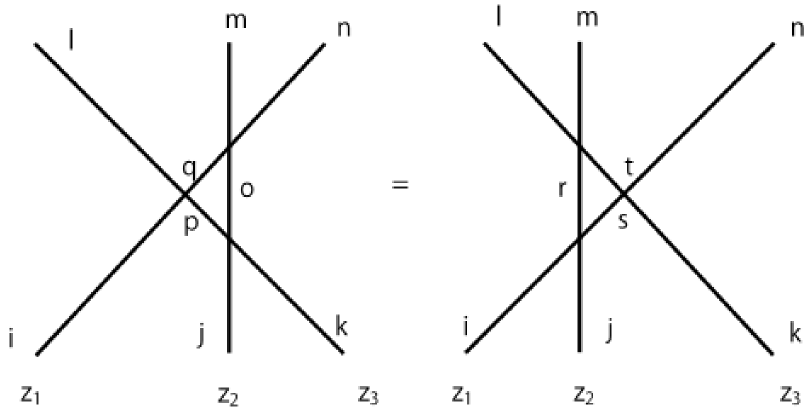
In the figure below we show the vertex on a pair of crossing worldlines and a particular vertex-network state (satisfying z-conservation) constructed out of such vertices.[1] Here we chose a specific combination of incoming versus outgoing z's, but we equivalently could have swapped z_1 and z_2 in the final line configuration. Similarly, in the right figure we chose to label horizontal and vertical lines as a whole.



In this figure we introduced some notation that accompanies a further restriction to the dynamics of our vertex theory. First off, we labeled each line with a *complex* "spectral parameter" z that is conserved at vertices. It turns out that the integrability of a vertex model requires such a z to be present [2, 10], and we will soon see that this extra complex quantum number (labeling the curves/lines in our vertex network) plays a crucial role in uncovering the algebraic structure of integrable vertex-network states. For our interpretation in terms of particles on a 1D manifold, we first note that integrability necessarily implies that the contact-interaction must be elastic [10], indicating that we could take a particle's momentum as its spectral parameter. Hence the dynamics is concentrated in the internal vector spaces V_{z_i} of worldlines, where the z_i subscript indicates that V belongs to a line with spectral parameter z_i, and at each collision the state is evolved according to a linear "R-matrix" $R(z_1, z_2) : V_{z_1} \otimes V_{z_2} \to V_{z_1} \otimes V_{z_2}$. In a particular basis, which was labeled with latin indices in the figure above, we may write this as $R^{kl}_{ij}(z_1, z_2)$.

Now, it turns out that in our vertex model, quantified by $R(z_1, z_2)$, the integrability condition reduces to a simple relation among lines in a vertex-network state [2], as displayed in the figure below (taken from [3]).

[1] Note that the exact same vertex structure arises for e.g. the "6-vertex model" [8] that e.g. models ice in the context of classical 2D statistical mechanics [9], in which case we envision a spin to occupy the middle of each line, with four spins interacting at each vertex. Furthermore, a combination of both interpretations holds for 1D quantum spin chains [6].



It amounts to the permission of sliding a worldline over the crossing of two other lines and is dubbed the "Yang-Baxter" relation. In our R-matrix language it reads:

$$R^{nm}_{qo}(z_1, z_2)R^{ql}_{ip}(z_1, z_3)R^{op}_{jk}(z_2, z_3) = R^{ml}_{rt}(z_2, z_3)R^{nt}_{sk}(z_1, z_3)R^{sr}_{ij}(z_1, z_2), \tag{1}$$

where we sum over repeated indices (in the figure above this represents summing over the internal states of the loop lines). In passing we note that the 6-vertex model mentioned earlier satisfies this relation [9], making it an integrable vertex model.

Instead of looking for solutions to the full Yang-Baxter eq. (3), we will instead consider a simpler and hence well-classified [11] variant that is defined in the presence of a perturbative parameter $\hbar$, so that we may expand $R(z_i, z_j) = I + \hbar r(z_i, z_j) + \mathcal{O}(\hbar^2)$. Here we suppressed the V-indices. We call $r_{ij} \equiv r(z_i, z_j)$ the "quasi-classical" R-matrix and substitute it in eq. (1) to obtain (after expanding the products):

$$\begin{aligned} &I + \hbar(r_{12} + r_{13} + r_{23}) + \hbar^2(r_{12}r_{13} + r_{12}r_{23} + r_{13}r_{23}) + \mathcal{O}(\hbar^3) = \\ &\qquad I + \hbar(r_{23} + r_{13} + r_{12}) + \hbar^2(r_{23}r_{13} + r_{23}r_{12} + r_{13}r_{12}) + \mathcal{O}(\hbar^3). \end{aligned} \tag{2}$$

We see that the $\mathcal{O}(\hbar^2)$ term yields the "classical" Yang-Baxter equation:

$$[r_{12}, r_{13} + r_{23}] + [r_{13}, r_{23}] = 0, \tag{3}$$

which is quadratic in the r_{ij} as opposed to eq. (1).

II. Elements of gauge theory

To see how these integrable vertex-network states connect to gauge theory, we observe that if we could somehow apply the well-known Wilson line and topological field theory (TFT) technologies to a 2D topological manifold Σ (in which the vertex-model lives) we could possibly reconstruct states such as e.g. shown in the figures above. In such a QFT interpretation we might expect the standard Feynman rules apply: assigning propagators to internal lines and interaction vertex factors at crossings. The complexity of integrability arises when we take into account that the integrable vertex model described in the previous section includes addition data: z, which prompts us to expand the gauge theory's base manifold to the product $\Sigma \times \mathbb{C}$. Hence we will be working with a 4D gauge theory.

As in [11] we will restrict ourselves to theories for which r_{ij} lives in $\mathfrak{g} \otimes \mathfrak{g}$, where $\mathfrak{g}$ is a semi-simple complex Lie algebra, so that for a particular basis t_a we may write:

$$r_{ij} = r_{ij}^{ab} t_a \otimes t_b. \tag{4}$$

As mentioned z is conserved along lines, which we transmute into the requirement that our gauge TFT must be translation invariant along the z-coordinate on $\mathbb{C}$ (the spectral parameter and complex coordinate are both called z). This is met by the gauge TFT action

$$S = \frac{1}{2\pi} \int_{\Sigma \times \mathbb{C}} dz \wedge \mathrm{CS}, \qquad \mathrm{CS} = \mathrm{tr}(A \wedge dA + \frac{2}{3} A \wedge A \wedge A), \tag{5}$$

where in the well-known Chern-Simons 3-form Lagrangian CS [12–16] we introduced the gauge field form $A = A_x dx + A_y dy + A_{\bar{z}} d\bar{z}$,[2] and where tr signifies the Killing form on $\mathfrak{g}$ normalized as $\mathrm{tr}(t_a t_b) = \delta_{ab}$. Note that $A_i \equiv A_i^a t_a$ takes value in $\mathfrak{g}$, implying that the corresponding Lie group G is the gauge group of our theory, and the $A_i^a(x, y, z, \bar{z})$ are understood as independent complex fields over the full $\Sigma \times \mathbb{C}$.

To study the gauge-properties of eq. (5), we note that $d\mathrm{CS}$ may be rewritten as the "second Chern form" $\mathrm{tr}(F \wedge F)$ [17] (with the field strength $F \equiv dA + A \wedge A$), which is of course gauge-invariant under $\delta_g : A \to gAg^{-1} + gd(g^{-1})$ (with $g \in G$) since F is. Consequently,

$$d\delta_g \mathrm{CS} = \delta_g d\mathrm{CS} = \delta_g \,\mathrm{tr}(F \wedge F) = 0 \quad \longrightarrow \quad \delta_g \mathrm{CS} = d\zeta, \tag{6}$$

showing that CS is invariant under homotopically trivial gauge-transformations (ζ denotes some boundary 2-form) and hence that S defines a proper gauge theory. Furthermore, the topological nature of the TFT described by S is encapsulated by the equation of motion, derived from the arbitrary first order variation δA (using that δA, being a derivation, satisfies the product rule on the wedges):

$$\delta S = 0 \quad \longrightarrow \quad \frac{1}{\pi} \int_{\Sigma \times \mathbb{C}} dz \wedge \mathrm{tr}(\delta A \wedge (dA + A \wedge A)) = 0 \quad \longrightarrow \quad F = 0, \tag{7}$$

which implies that all local gauge-invariant observables vanish, as these are of course necessarily constructed out of the gauge-invariant F. We conclude that the gauge field has trivial local dynamics as required.

[2] Note that $A_z dz$ is excluded due to the required translation invariance along z, which motivated the dz in eq. (5). Moreover, we could have equivalently interchanged the roles of z and $\bar{z}$.

This tells us that if we want to extract any physics from eq. (5), we must consider non-local gauge-invariants, which allow us to define non-trivial quantum numbers that relate to the base manifold topology. A simple example concerns the winding number of loops on S^1, which gives rise to an instanton sector for e.g. the free boson [13, 18], but in our case we will need the more sophisticated Wilson loop apparatus. In particular, we will consider the behavior of A_i as we transverse a loop L in $\Sigma \times \mathbb{C}$, which we choose to quantify with the "Wilson loop" W:

$$W = \mathrm{tr}_\rho \, \mathcal{P} \exp\big(\oint_L A_i(x, y, z, \bar{z}) dx^i \big), \tag{8}$$

where $\mathcal{P}$ denotes path-ordering and tr_ρ once again signifies the Killing form. It can be shown that the argument ξ of tr_ρ in eq. (8), the closed Wilson line, transforms in some representation ρ of our gauge algebra $\mathfrak{g}$ [19] (in which we hence also defined tr_ρ), i.e. $\xi \to U\xi U^{-1}$ for $U \in \rho$.[3] Hence the cyclicity of tr implies that W is gauge-invariant.

Crucially, since by definition our gauge field A_i does not contain A_z, we see that eq. (8) is trivial for $L \in \mathbb{C}$ with x, y constant, as only $\bar{z}$ is allowed to vary. In other words, we are led to consider $L \in \Sigma$ with z constant, which corresponds to a loop labeled by z living in a 2D topological space (so that only $i = 1, 2$ contribute). These are precisely the features that our lines in the vertex-network states from section I possessed! Accordingly, instead of using W to label homotopically inequivalent loops, as in the S^1 winding example, we use it to define a quantum number that represents the amount of nodes in a network of Wilson loops.

It turns out that eq. (5) and hence all the structure that followed (and will follow) can be generated by deforming $\mathcal{N} = 1$ supersymmetric gauge theory and any $\mathcal{N} = 2$ supersymmetric field theory [1, 2], by using the supercharges to induce the invariance structure on $\Sigma \times \mathbb{C}$.

III. Quasi-classical R-matrix from perturbative gauge theory

To quantify the correspondence between 2D integrable vertex models and 4D gauge theory sketched at the end of last section, we need to consider the quantum properties of our gauge TFT in perturbation theory, to which end we quantize eq. (5) via the Euclidean path integral:

$$\langle \mathcal{O} \rangle = \int \mathcal{D}A \mathcal{O} \exp\left(-\frac{S}{\hbar}\right) \Big/ \int \mathcal{D}A \exp\left(-\frac{S}{\hbar}\right), \tag{9}$$

[3] Specifically, (local) gauge transforming a Wilson line reduces to separate G-transforms $U, U' \in \rho$ on its ends [17], so that by forming a loop we find that ξ transforms in ρ.

where $\mathcal{O}$ is an arbitrary operator insertion. We consider the loop-counting[4] parameter $\hbar$ to be variable, so that we may take the semi-classical limit $\hbar \to 0$ upon finishing our calculation, in which case it coincides with its namesake: the perturbative parameter $\hbar \to 0$ that was used to define the quasi-classical R-matrix in section I.

As per usual, one of the main building blocks of our perturbation theory will be the Euclidean propagator described by eq. (5), which we find from the corresponding free Lagrangian:

$$\mathrm{tr}(A \wedge dA) = \epsilon^{ijk}\,\mathrm{tr}(A_i \partial_j A_k) = \epsilon^{ijk} A_i^a \partial_j A_k^b\,\mathrm{tr}(t_a t_b) = \delta_{ab}\epsilon^{ijk} A_i^a \partial_j A_k^b, \tag{10}$$

by inverting its integral kernel [20] (reinserting the $1/2\pi$ coefficient from S and in the third line an infinitesimal $i\epsilon$ to avoid the IR divergence):[5]

$$\langle A_i^a(x, y, z, \bar{z}) A_k^b(x', y', z', \bar{z}')\rangle = \left[\frac{\delta_{ab}}{2\pi}\epsilon^{ijk}\partial_j\right]^{-1} \tag{11}$$

$$= \int \frac{d^4k}{(2\pi)^4}\left[\frac{\delta_{ab}}{2\pi}\epsilon^{ijk}k_j\right]^{-1} \exp\left(-ik_\mu(x^\mu - x'^\mu)\right) \tag{12}$$

$$= \int \frac{d^4k}{(2\pi)^4}\frac{2\pi\delta^{ab}\epsilon_{ijk}k^j}{2(k^2 - i\epsilon)} \exp\left(-ik_\mu(x^\mu - x'^\mu)\right), \tag{13}$$

where $k_\mu x^\mu = \delta^{\mu\nu}k_\mu k_\nu$.[6] In an effort to simplify this propagator, we note the well-known $d = 1 + 3$ massless scalar Euclidean propagator result [20]:

$$\langle \phi(\vec{x})\phi(\vec{x}')\rangle = \int \frac{d^4k}{(2\pi)^4}\frac{\exp\left(-ik_\mu(x^\mu - x'^\mu)\right)}{k^2 - i\epsilon}, \tag{14}$$

so that we may write:

$$\langle A_i^a(\vec{x}) A_k^b(\vec{x}')\rangle = i\pi\delta^{ab}\epsilon_{ijk}\partial^j\langle\phi(\vec{x})\phi(\vec{x}')\rangle. \tag{15}$$

To perform perturbative calculations in real-space we require a metric, as to define a norm $|\vec{x}|$, to which end we first specialize to $\Sigma = \mathbb{R}^2$. Since we will be computing corrections to a crossing, we find that this causes no loss of generality, because the diffeomorphism invariance on Σ allows us to "zoom in" on a crossing (i.e. blow up the metric scale) until we decide that the curvature approximately vanishes at the crossing point. This is actually what allows our local R-matrix picture to survive quantization, since by zooming in the inter-crossing corrections will eventually become negligible compared to intra-crossing corrections. Note that the decrease in curvature upon zooming in simply follows from the definition of a manifold.

[4]This property follows from the semi-classical expansion of eq. (9) [20].

[5]The expectation value is with respect to the vacuum.

[6]Note that Latin indices run over $x, y, \bar{z}$ while Greek indices run over $x, y, z, \bar{z}$.

We choose to work with a flat metric on $\mathbb{R}^2 \times \mathbb{C}$:

$$d|\vec{x}|^2 = g_{\mu\nu} dx^\mu dx^\nu = dx^2 + dy^2 + dz d\bar{z}, \tag{16}$$

so that upon writing eq. (14) as [20]:

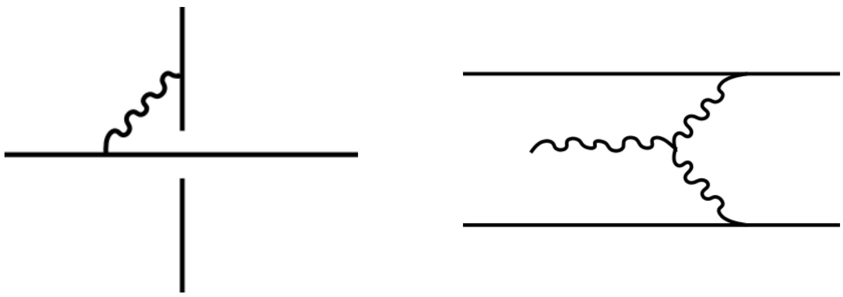
$$\langle \phi(\vec{x}) \phi(\vec{x}') \rangle = \lim_{m \to 0} \frac{-im}{4\pi^2 |\vec{x} - \vec{x}'|} K_1(m|\vec{x} - \vec{x}'|) = \frac{-i}{4\pi^2 |\vec{x} - \vec{x}'|^2}, \tag{17}$$

where K_1 is a modified Bessel function and where we used $\lim_{x \to 0} K_1(x) = 1/x$ [20], we finally get

$$\langle A_i^a(x, y, z, \bar{z}) A_k^b(x', y', z', \bar{z}') \rangle = \delta^{ab} \epsilon_{ijk} \partial^j \frac{1}{4\pi |\vec{x} - \vec{x}'|^2}. \tag{18}$$

Note that we have completely ignored the quadratic gauge-fixing procedure which is usually required to define a well-defined gauge propagator [19, 20], which is justified in light of the free CS-term containing but a single derivative (see eq. 5). Of course, we must still fix a gauge, which we choose to be $\partial_x A_x + \partial_y A_y + 4\partial_z A_{\bar{z}} = 0$,[7] as a result of which eq. (18) must satisfy $\partial_x \langle A_x^a A_i'^b \rangle + \partial_y \langle A_y^a A_j'^b \rangle + 4\partial_z \langle A_{\bar{z}}^a A_k'^b \rangle = 0$. From eq. (18) we see that this gauge constraint is easily incorporated by slightly modifying the normalization of some propagator components, as will become explicit below. This is all the gauge-fixing we will need.

There is one more thing left before we are finally able to calculate some interesting perturbative effects. We import the gauge boson-Wilson line vertex factor from regular non-Abelian gauge theory [21]: $-it_a \hat{n}$, with t_a in the representation of A and the line's internal space, and where $\hat{n}$ selects the gauge field component orthogonal to the line. With these tools at hand, let us now consider the left diagram below.



This diagram shows a one-loop quantum correction to a (zoomed-in) crossing, i.e. a correction to the R-matrix, corresponding to gauge boson exchange. The displayed ordering of the crossing is based on the difference in $(z, \bar{z})$ base points of the crossing Wilson loops, as a result of which these loops do not cross in $\Sigma \times \mathbb{C}$, i.e. there is no loop-loop vertex factor. The horizontal loop lies along the x-axis at $(y_2, z_2, \bar{z}_2)$ and the vertical loop along the y-axis at $(x_1, z_1, \bar{z}_1)$, over which we must integrate the vertices to take into account the whole

[7] This is the Lorentz gauge, in the sense that consequently $\Box A_i = 0$ [3].

configuration space of the internal gauge boson. Note that we used the diffeomorphism invariance on Σ to represent a Wilson loop as an infinite line.

Combining all of the above and using the usual Feynman rules, we find the following amputated amplitude (expanding the derivative in the (x, y)-component of eq. (18)):

$$\begin{aligned} i\mathcal{M} &= \hbar(-it_a) \otimes (-it_b) \int dxdy \cdot -i\langle A^a(x_1, y, z_1, \bar{z}_1) A^b(x, y_2, z_2, \bar{z}_2)\rangle \\ &= i\hbar t_a \otimes t^a \int_{-\infty}^{\infty} dxdy \frac{2(\bar{z}_1 - \bar{z}_2)}{2\pi} \frac{1}{((x_1 - x)^2 + (y - y^2) + (z_1 - z_2)(\bar{z}_1 - \bar{z}_2))} \end{aligned}$$

where $\mathcal{M} \in \mathfrak{g} \otimes \mathfrak{g}$ due to the product of Wilson line G-representations induced by the gauge boson, and where the factor $\hbar$ is induced by the single loop. Performing the integrals yields:

$$\mathcal{M} = \frac{\hbar t_a \otimes t^a}{z_1 - z_2}, \tag{19}$$

which is the "rational" quasi-classical R-matrix solution [3, 6, 11], i.e. $r_R(z_1, z_2) = \mathcal{M}/\hbar$.

Hence, we have found a solution of eq. (3) by considering a gauge theory! In fact, it turns out that this $\mathcal{O}(\hbar)$ term determines the rational R-matrix to all orders [22]. For $\mathfrak{g} = \mathfrak{sl}_2$, with the Wilson lines transforming in the fundamental representation, it can be shown that the rational R-matrix encodes the Heisenberg XXX spin-chain [2].

IV. Fusing lines: enhanced algebra

Next we consider the second diagram drawn above, which contains a new ingredient: the CS interaction 3-vertex. From eq. (5) we find the corresponding Feynman rule vertex factor η (including the symmetry factor):

$$\frac{dz}{2\pi} \wedge \epsilon^{ijk} \operatorname{tr}\left(\frac{2}{3} A_i A_j A_k\right) \quad \longrightarrow \quad \eta = \frac{i}{2\pi} \epsilon^{ijk} \operatorname{tr}(t_a t_b t_c) dz = \frac{i}{2\pi} \epsilon^{ijk} f_{abc} dz, \tag{20}$$

where $\operatorname{tr}(t_a t_b t_c) = f_{abc}$ follows by using the definition $[t_a, t_b] = f_{ab}^c t_c$, the Killing form $\operatorname{tr}(t_a t_b) = \delta_{ab}$ (and consequently $f_{abc} = f_{[abc]}$), and the cyclicity of the trace. Moreover, the upper line lies along the x-axis at $y = \epsilon$, with the boson-Wilson vertex parametrized by x_1, and the lower line at $y = 0$, parametrized by x_2; this time both lines are supported at the $(z, \bar{z}) = (0, 0)$. The CS vertex is parametrized by $(x, y, z, \bar{z}) \in \mathbb{R}^2 \times \mathbb{C}$. Again, we will integrate over the one-loop parameters x_1, x_2, x, y, z and $\bar{z}$ to take into account its full configuration space.

For convenience we now define the gauge field propagator 2-form:

$$P^{ab}(\vec{x}_1, \vec{x}_2) = \delta^{ab} P = \frac{\delta^{ab}}{2\pi} \frac{(x_1 - x_2)dy \wedge d\bar{z} + (y_1 - y_2)d\bar{z} \wedge dx + 2(\bar{z}_1 - \bar{z}_2)dx \wedge dy}{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)(\bar{z}_1 - \bar{z}_2)}, \tag{21}$$

so that we drop the Latin indices in eq. (20). In this language the amputated amplitude corresponding to the right one-loop diagram reads:

$$i\mathcal{M}' = \hbar t^a \otimes t^b \int_{-\infty}^{\infty} dx_1 dx_2 \int_{\mathbb{R}^2\times\mathbb{C}} -iP(x, x_1; y, \epsilon; z, 0; \bar{z}, 0) \wedge A^c \wedge \frac{i}{3\pi} f_{abc} dz \wedge -iP(x, x_2; y, 0; z, 0; \bar{z}, 0),$$

where $A^c = A^c(x, y, z, \bar{z})$ denotes the external gauge field that is attached to the 3-vertex (and which we do not amputate). Now, we mentioned earlier that a Wilson line couples orthogonally to A_i, implying that the lines under consideration could couple to A_y and $A_{\bar{z}}$. Since the corresponding gauge bosons are internal, we sum over their states; from eq. (21) we have the sum of orthogonal P^{ab} components:

$$P^{\perp} = -\frac{1}{4\pi} \frac{d\bar{z} \wedge dx\partial_y + 4dx \wedge dy\partial_z}{(x - x_1)^2 + (y - \epsilon)^2 + z\bar{z}}, \tag{22}$$

which we substitute into $i\mathcal{M}'$ above and subsequently integrate over x_1, yielding a factor:

$$P' \equiv -\frac{1}{4\pi} (\partial_y d\bar{z} + 4\partial_z dy)\wedge dx \int_{-\infty}^{\infty} dx_1 \frac{1}{(x - x_1)^2 + (y - \epsilon)^2 + z\bar{z}} = -\frac{1}{4}(\partial_y d\bar{z} + 4\partial_z dy) \frac{1}{\sqrt{y^2 + z\bar{z}}},$$

where we used $dy \wedge dx = -dx \wedge dy$ and where we set $dx = 0$ in the second equality since x was eliminated. Repeating this for the propagator containing x_2, we find that $\mathcal{M}'$ with the boson-Wilson couplings integrated out becomes

$$i\mathcal{M}' = -\frac{\hbar}{2\pi} f_{abc} t^a \otimes t^b \int_{\mathbb{R}^2\times\mathbb{C}} P'(y, \epsilon; z, 0; \bar{z}, 0) \wedge A^c(x, y, z, \bar{z}) \wedge dz \wedge P'(y, 0; z, 0; \bar{z}, 0).$$

Let us consider what happens when $\epsilon \to 0$, i.e. when the two Wilson lines fuse[8] into a new Wilson line that lies along the x-axis at $y = 0$. Due to the antisymmetry of the wedge product and the fact that P' is a 1-form, it appears as if the integrand simple vanishes in this limit. However, from P' above we see that it blows up at $y = 0 = z$, so that this argument holds everywhere but here. In fact, it turns out that [3]:

$$\lim_{\epsilon\to 0} P'(y, \epsilon; z, 0; \bar{z}, 0) \wedge dz \wedge P'(y, 0; z, 0; \bar{z}, 0) = -i\pi\partial_z \delta^{(3)}(y, z, \bar{z}), \tag{23}$$

and consequently we find the possibly non-vanishing

$$\mathcal{M}' = -\frac{\hbar}{2} f_{abc} t^a \otimes t^b \int_{\mathbb{R}} dx \partial_z A^c(x, 0, z, 0)\Big|_{z=0}, \tag{24}$$

to which end we moved P' past A^c and integrated by parts. This result tells us that two Wilson loops are indeed able to fuse, since it represents a Wilson loop along the x-axis at $y = 0$, coupled to $\partial_z A^c$ instead of the usual A^c.

[8] Obviously, this is hand-waving at its finest, but it turns out that this fusing of lines be made rigorous via an operator product expansion (OPE) construction, familiar from conformal field theory [23].

To interpret this we are led to consider a more elaborate Wilson operator, where instead of a single loop at (for convenience) base point $(0,0) \in \mathbb{C}$ we consider an expansion in z with Wilson loops, i.e. elements of a $\mathfrak{g}$ representation, as coefficients. More specifically, this new "loop" has internal states that transform according to a representation of the extended algebra $\mathfrak{g}[[z]] \equiv \prod_{n\geq 0} \mathfrak{g} \otimes z^n$, with basis $\mathbf{t}_{a,n} \equiv t_a z^n$ satisfying $[\mathbf{t}_{a,n}, \mathbf{t}_{b,m}] = f_{ab}^c \mathbf{t}_{c,n+m}$ (where $t_a \in \mathfrak{g}$ and $n \geq 0$). To probe its coupling to the gauge field we define the z-dependence of A_i according to the expansion:

$$A_i(x,y,z,0) = \sum_{k\geq 0} \frac{z^k}{k!} \partial_z^k A_i(x,y,z,0)\bigg|_{z=0}, \tag{25}$$

so that the boson-Wilson interaction vertex $t_a A_i \hat{n}$ [21] yields the $k = 1$ interaction term: $t_a z \partial_z A_i = \mathbf{t}_{a,1} \partial_z A_i$. Such a coupling proportional to z is foreign to the conventional Wilson loops from before, since those are defined for constant z. Integrating the corresponding vertex factor over a Wilson line along the x-axis, at $y = 0$, evidently yields eq. (24) if we accept that

$$\mathbf{t}_{a,1} = -\frac{\hbar}{2} f_{abc} t^a \otimes t^b. \tag{26}$$

Therefore, if we want to include the fusing of $\mathfrak{g}$-valued Wilson lines into our theory, we find that closure of the corresponding fusion algebra requires us to introduce $\mathfrak{g}[[z]]$-valued Wilson lines.

The algebraic structure gets even more complicated when we realize that the upgrade to $\mathfrak{g}[[z]]$ only suffices for classical Wilson lines, since it is easily confirmed that the fusing quantum correction in eq. (26) does not satisfy $[\mathbf{t}_{a,1}, \mathbf{t}_{a,1}] = \mathbf{t}_{a,2}$. We could remedy this by adding a few terms to the commutation relations of (the universal enveloping algebra of) $\mathfrak{g}[[z]]$ [3], thereby giving us the "Yangian" deformation[9], which is known to be the algebra underlying the *full* rational R-matrix solution [11].

V. Concluding remarks

During our journey we have seen that gauge theory is remarkably successful at incorporating structures found in integrable vertex models. Obviously, there is much more to integrability than just the calculated rational r-matrix and fusing corrections, but it does seem promising that the Wilson loop machinery is able to reproduce the Yangian and rational r-matrix in such a natural and cohesive manner.

Indeed, it turns out that this correspondence is not peculiar to the rational solution. In particular, by interchanging $\mathbb{C}$ with a more general Riemann surface, and making some

[9] It turns out that the $\mathcal{O}(\hbar)$ correction in eq. (26) determines the Yangian to all orders [3, 4].

slight modifications to the formalism, it is possible to reproduce the trigonometric and elliptic quasi-classical solutions [3, 4], and the corresponding "quantum affine" respectively "elliptic" algebras that underly the full R-matrix solutions [6, 11]. As before, this is done using Wilson operators.

At this point the correspondence is still not close to being exhausted, since e.g. cancellation of the anomalous gauge behavior of quantized Wilson loops yields all sorts of interesting (well-known) constraints in the corresponding integrable vertex model [3]. This and much more can be found in [3, 4], including a detailed account of the material covered in this digest.

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