1. Brownian motion

d dimensions, square lattice

basis vectors $\hat{\underset{\sim}{n}}_\mu$ $\quad \mu = 1, \dots, d$

Pt on lattice: $\quad \underset{\sim}{r} = a \sum_{\mu=1}^{d} n^\mu \hat{\underset{\sim}{n}}_\mu$

a: lattice cst; $n^\mu \in \mathbb{Z}$

Random walker obeys 2 rules:

- 1: at each time step δt, RW takes one step
- 2: dirⁿ of step is uniformly distributed over all possibilities.

Need: $P_{\underset{\sim}{r}_1, t_1 | \underset{\sim}{r}_0, t_0}$

conditional proba of going from $\underset{\sim}{r}_0$ to $\underset{\sim}{r}_1$ in time $t_1 - t_0$

Sum rule: $\sum_{\underset{\sim}{r}_1} P_{\underset{\sim}{r}_1 t_1 | \underset{\sim}{r}_0 t_0} = 1 \quad \forall\, t_1$

Simple initial state: $P_{\underset{\sim}{r}, t_0 | \underset{\sim}{r}_0, t_0} = \delta_{\underset{\sim}{r}, \underset{\sim}{r}_0}$

Discrete time evolution:

$$P_{\underset{\sim}{r}_1, t_1+\delta t \,|\, \underset{\sim}{r}_0, t_0} = \frac{1}{2d} \sum_{\sigma=\pm} \sum_{\mu=1}^{d} P_{\underset{\sim}{r}+a\sigma\hat{m}_\mu, t_1 \,|\, \underset{\sim}{r}_0, t_0}$$

DTEE1

Use discrete Laplacian:

$$\nabla_a^2 f_{\underset{\sim}{r}} \equiv \frac{1}{a^2} \sum_{\mu=1}^{d} \left[f_{\underset{\sim}{r}+a\hat{m}_\mu} + f_{\underset{\sim}{r}-a\hat{m}_\mu} - 2 f_{\underset{\sim}{r}} \right]$$

$$\lim_{a\to 0} \nabla_a^2 f_{\underset{\sim}{r}} = \nabla^2 f(\underset{\sim}{r})$$

Time evol^{n}:

$$P_{\underset{\sim}{r}_1, t_1+\delta t \,|\, \underset{\sim}{r}_0, t_0} - P_{\underset{\sim}{r}_1, t_1 \,|\, \underset{\sim}{r}_0, t_0} = \frac{a^2}{2d} \nabla_a^2 P_{\underset{\sim}{r}_1, t_1 \,|\, \underset{\sim}{r}_0, t_0}$$

In continuum limit:

because

$$\left[\partial_t - D \nabla^2 \right] P(\underset{\sim}{r}, t) = 0$$

Lim scaling $\dfrac{a^2}{2d\,\delta t} \equiv D$

Sol^{n}: Fourier

For ∞ lattice:

$$f_{\underset{\sim}{r}} = a^d \int_{-\pi/a}^{\pi/a} \frac{d^d \hat{k}}{(2\pi)^d} e^{i \underset{\sim}{k} \cdot \underset{\sim}{r}} f(\underset{\sim}{k})$$

$$f(\underset{\sim}{k}) = \sum_{\underset{\sim}{r}} e^{-i \underset{\sim}{k} \cdot \underset{\sim}{r}} f_{\underset{\sim}{r}}$$

Initial cond^{n}: $P_{\underset{\sim}{r}, t_0 | \underset{\sim}{r}_0, t_0} = \delta_{\underset{\sim}{r}, \underset{\sim}{r}_0}$

$$\rightarrow \quad P_{t_0 | r_0, t_0}(\underset{\sim}{k}) = e^{-i \underset{\sim}{k} \cdot \underset{\sim}{r}_0}$$

TEE:

$$\sum_{\underset{\sim}{r}} e^{-i \underset{\sim}{k} \cdot \underset{\sim}{r}} P_{\underset{\sim}{r}, t_1 + \delta t} = \sum_{\underset{\sim}{r}} e^{-i \underset{\sim}{k} \cdot \underset{\sim}{r}} \frac{1}{2d} \sum_{\sigma = \pm} \sum_{\mu=1}^{d} P_{\underset{\sim}{r} + a\sigma \hat{\underset{\sim}{n}}_\mu, t_1}$$

$$\rightarrow P_{\underset{\sim}{k}, t_1 + \delta t} = \frac{1}{2d} \sum_{\mu=1}^{d} \sum_{\sigma = \pm} \sum_{\underset{\sim}{r}} e^{-i \underset{\sim}{k} \cdot (\underset{\sim}{r} - a\sigma \hat{\underset{\sim}{n}}_\mu)} P_{\underset{\sim}{r}, t_1}$$

$$= \left[\frac{1}{d} \sum_{\mu=1}^{d} \cos(k^\mu a) \right] P_{\underset{\sim}{k}, t_1}$$

This is of form

$$P_{\underset{\sim}{k}, t_1 + \delta t} = [\text{some } \#] \, P_{\underset{\sim}{k}, t_1}$$

$$\left(\text{N.B.: } k^\mu = \underset{\sim}{k} \cdot \hat{\underset{\sim}{n}}_\mu \right)$$

So:
$$P_{\underset{\sim}{k}, t_1} = \left[\frac{1}{d} \sum_\mu \cos k^\mu a \right]^{\frac{t_1 - t_0}{\delta t}} P_{k, t_0}$$

$\rightarrow$ __no__ approx^{n}! Exact.

FT back to $\underset{\sim}{x}$ space:

$$P_{\underset{\sim}{x}, t_1 | \underset{\sim}{x}_0, t_0} = a^d \int_{-\pi/a}^{\pi/a} \frac{d^d \underset{\sim}{k}}{(2\pi)^d} e^{i \underset{\sim}{k}\cdot(\underset{\sim}{x}_1 - \underset{\sim}{x}_0)} \left[\frac{1}{d}\sum_{\mu=1}^{d} \cos k^\mu a\right]^{\frac{t_1-t_0}{\delta t}}$$

Continuum limit: $\frac{t_1-t_0}{\delta t} \to$ very large

$\to$ keep only momenta k^μ near 0

$$\left[\frac{1}{d}\sum_{\mu} \cos k^\mu a\right]^{\frac{t_1-t_0}{\delta t}} = \left[1 - \frac{a^2}{2d} \underset{\sim}{k}^2 + \dots\right]^{\frac{t_1-t_0}{\delta t}} \to e^{-(t_1-t_0)\frac{a^2}{2d\delta t} \underset{\sim}{k}^2}$$

$$\lim_{n\to\infty} \left[1 + \frac{c}{n}\right]^n = e^c$$

$$\left[1 - \frac{\frac{a^2}{2d} k^2 \left(\frac{t_1-t_0}{\delta t}\right)}{\frac{t_1-t_0}{\delta t}}\right]^{\frac{t_1-t_0}{\delta t}}$$

Scaling limit:

$\delta t \to 0, \quad a^2 \to 0$

s.t. $\frac{a^2}{\delta t}$ is finite, so $\frac{a^2}{2d\delta t} = D$ finite

$t_1 - t_0$ finite

Prob per unit volume:

$$p(\underset{\sim}{x}_1, t_1 | \underset{\sim}{x}_0, t_0) = \lim_{\text{scaling}} a^{-d} P_{\underset{\sim}{x}_1, t_1 | \underset{\sim}{x}_0, t_0}$$

$$=$$

$$\cdot\ p(\underset{\sim}{r}_1, t_1 \mid \underset{\sim}{r}_0, t_0) = \lim_{\text{scaling}} a^{-d} P_{\underset{\sim}{r}_1 t_1 \mid \underset{\sim}{r}_0, t_0}$$

$$= \int_{-\infty}^{\infty} \frac{d^d \underset{\sim}{k}}{(2\pi)^d}\, e^{-(t_1 - t_0) D \underset{\sim}{k}^2 + i \underset{\sim}{k} \cdot (\underset{\sim}{r}_1 - \underset{\sim}{r}_0)}$$

$$= \frac{1}{[4\pi D(t_1 - t_0)]^{\frac{d}{2}}} \exp\left[-\frac{|\underset{\sim}{r}_1 - \underset{\sim}{r}_0|^2}{4D(t_1 - t_0)} \right]$$

$$(\partial_t - D\nabla^2)\, p(\underset{\sim}{r}, t) = 0$$

$$\int d^d \underset{\sim}{r}\; p(\underset{\sim}{r}, t) = 1 \quad \forall t$$

Nice property: composition

$$P(\underset{\sim}{r}_2, t_2 \mid r_0, t_0) = \int d^d \underset{\sim}{r}_1\; P(\underset{\sim}{r}_2, t_2 \mid \underset{\sim}{r}_1, t_1)\, P(\underset{\sim}{r}_1, t_1 \mid r_0, t_0)$$

$$t_2 > t_1 > t_0$$

Proof: exercise.

Rare events: what is proba of diffusing a distance $\propto t$?

Simple $q^{\underline{n}}$: how much time does RW spends on a given pt $\underset{\sim}{r}_1$?

$$\sum_{n=0}^{\infty} P_{\underset{\sim}{r}_1, t_0+n\delta t | \underset{\sim}{r}_0, t_0} \equiv G_{\underset{\sim}{r}_1-\underset{\sim}{r}_0}$$

$$\sum_{n=0}^{\infty} a^d \int_{-\pi/a}^{\pi/a} \frac{d^d k}{(2\pi)^d} e^{i\underset{\sim}{k}\cdot(\underset{\sim}{r}_1-\underset{\sim}{r}_0)} \left[\frac{1}{d}\sum \cos k^\mu a\right]^n$$

$$\sum_{n=0}^{\infty} \alpha^n = \frac{1}{1-\alpha}$$

so
$$G_{\underset{\sim}{r}_1-\underset{\sim}{r}_0} = a^d \int_{-\pi/a}^{\pi/a} \frac{d^d \underset{\sim}{k}}{(2\pi)^d} \frac{e^{i\underset{\sim}{k}\cdot(\underset{\sim}{r}_1-\underset{\sim}{r}_0)}}{1-\frac{1}{d}\sum_\mu \cos k^\mu a}$$

We'll call this the Green's $f^{\underline{n}}$.

It obeys the eq $^{\underline{n}}$

$$G_{\underset{\sim}{r}_1-\underset{\sim}{r}_0} = \delta_{\underset{\sim}{r}_1,\underset{\sim}{r}_0} + \frac{1}{2d}\sum_\mu \left[G_{\underset{\sim}{r}_1+a\hat{\underset{\sim}{m}}_\mu,\underset{\sim}{r}_0} + G_{\underset{\sim}{r}_1-a\hat{\underset{\sim}{m}}_\mu,\underset{\sim}{r}_0}\right]$$

$$\rightarrow \quad -\nabla_a^2 G_{\underset{\sim}{r}_1-\underset{\sim}{r}_0} = \frac{2d}{a^2}\delta_{\underset{\sim}{r}_1,\underset{\sim}{r}_0}$$

Divergences.

Focusing on region $|k| \sim 0$:

$$\int d^d \underset{\sim}{k} \frac{e^{i\underset{\sim}{k}\cdot\underset{\sim}{r}}}{\underset{\sim}{k}^2}$$

If $d>2$:

For $d=2$: $\int_0^{2\pi} d\theta \int_0^{\infty} dk\, k \frac{e^{ik\cdot r}}{k^2} \sim$ log diverg.

1st way to deal with div: take it away

Define a 'subtracted' Green's f^n,

$$G^S_{\underset{\sim}{r}_1-\underset{\sim}{r}_0} = G_{\underset{\sim}{r}_1-\underset{\sim}{r}_0} - G_0 = a^d \int \frac{d^d \underset{\sim}{k}}{(2\pi)^d} \frac{e^{i\underset{\sim}{k}\cdot(\underset{\sim}{r}_1-\underset{\sim}{r}_0)} - 1}{1 - \frac{1}{d}\sum \cos k_\mu}$$

Ex: in 1d,

$$G^S_{\underset{\sim}{r}_1-\underset{\sim}{r}_0} = -\frac{|\underset{\sim}{r}_1-\underset{\sim}{r}_0|}{a}$$

2nd way: regularization

- Rule 3: at each time interval, RW quits the game with prob. η (small η)

Sol^n: earlier sol^n, dressed with factor

$$[1-\eta]^{\frac{t_1-t_0}{\delta t}}$$

$$G_{\underset{\sim}{r}_1-\underset{\sim}{r}_0} = \sum_{n=0}^{\infty} (1-\eta)^n P_{\underset{\sim}{r}_1, t_0+n\delta t}$$

So

$$G = \int d^d k \, \frac{e^{i\underset{\sim}{k}\cdot\underset{\sim}{r}}}{1 - \frac{(1-\eta)}{d}\sum \cos}$$

$\downarrow$

obeys the eq^n:

$$\left[-\nabla_a^2 + m^2\right] G_{\underset{\sim}{r}_1-\underset{\sim}{r}_0} = \frac{2d}{a^2(1-\eta)} \delta_{\underset{\sim}{r}_1,\underset{\sim}{r}_0}$$

$$\frac{2d}{a^2}\frac{\eta}{1-\eta}$$

Continuum limit: $\left[-\nabla^2 + m^2\right] G_{\underset{\sim}{r}} = \delta(\underset{\sim}{r})$

sol^n $G_m(\underset{\sim}{r}) = \int d^d \underset{\sim}{k} \, \frac{e^{i\underset{\sim}{k}\cdot\underset{\sim}{r}}}{k^2+m^2}$

Explicit sol$^{\underline{ns}}$ for $d=1,2,3$: see notes

$d=1$: $$G_m(r) = \frac{e^{-m|r|}}{2m}$$

$d=2$: see notes

$d=3$: $$G_m(r) = \frac{e^{-m|\underline{r}|}}{4\pi|\underline{r}|}$$

Towards a path integral

Reinterpretation:

$$P_{\underline{r}_1,t_1|r_0,t_0} = \frac{\#\text{ paths from } \underline{r}_0 \text{ to } \underline{r}_1 \text{ in time } t_1-t_0}{\text{total }\#\text{ of paths in time } t_1-t_0}$$

In scaling limit, Recall

$$P(\underline{r}_f,t_f|\underline{r}_i,t_i) = \frac{1}{[4\pi D(t_f-t_i)]^{\frac{d}{2}}} e^{-\frac{|\underline{r}_f-\underline{r}_i|^2}{4D(t_f-t_i)}}$$

For given t_f & t_i, 'slice' the interval into N pieces (& let $N\to\infty$ eventually)

Use composition property: $t_n = t_i + n\Delta t$, $\Delta t = \frac{t_f-t_i}{N}$

$$P(\underline{r}_f,t_f|\underline{r}_i,t_i) = \int \prod_{n=1}^{N-1} d^d\underline{r}_n \, P(\underline{r}_f,t_f|\underline{r}_{N-1},t_{N-1}) \times$$
$$\times P(\underline{r}_{N-1},t_{N-1}|\underline{r}_{N-2},t_{N-2}) \cdots P(\underline{r}_1,t_1|\underline{r}_i,t_i)$$

For a small interval Δt, we have (letting $\Delta t \to 0$)

$$P(\underset{\sim}{r}_{n+1}, t_{n+1} \,|\, \underset{\sim}{r}_n, t_n) \to \frac{1}{[4\pi D \Delta t]^{\frac{d}{2}}} \exp\left\{ - \frac{|\underset{\sim}{r}_{n+1} - \underset{\sim}{r}_n|^2}{4D\Delta t} \right\}$$

Write $\dfrac{\underset{\sim}{r}_{n+1} - \underset{\sim}{r}_n}{\Delta t} \to \dfrac{d}{dt}\underset{\sim}{r}(t)$

$\to$ our proba becomes a <u>path integral</u>:

$$P(\underset{\sim}{r}_f, t_f \,|\, \underset{\sim}{r}_i, t_i) = \int_{\substack{\underset{\sim}{r}(t_i) = \underset{\sim}{r}_i \\ \underset{\sim}{r}(t_f) = \underset{\sim}{r}_f}} \mathcal{D}\underset{\sim}{r}(t) \; \exp\left\{ -\frac{1}{4D} \int_{t_i}^{t_f} dt \left| \frac{d\underset{\sim}{r}(t)}{dt} \right|^2 \right\}$$

Path integral measure:

$\mathcal{D}$: see eq$^{\underline{n}}$ 77 in notes.