The Ising model

Sites on a lattice; at each site,

variable $s_i = \pm 1$ (i ← site index)

General $H = -B\sum_{i} s_{i} + \frac{1}{2}\sum_{i,j} J_{ij}\, s_{i} s_{j}$

Simplest choice:

$$J_{ij} = \begin{cases} J & i,j \text{ are n.n.} \\ 0 & \text{otherwise} \end{cases}$$

Want

$$Z = \sum_{\{s_i\}} \exp\left\{ \beta B \sum_{i} s_{i} - \frac{\beta}{2} \sum_{i,j} J_{ij}\, s_{i} s_{j} \right\}$$

1d. Choose $J \equiv -\varepsilon$ $\varepsilon > 0$

Lattice of N sites, use PBCs

$s_{i+N} \equiv s_i$

$$H = \sum_{i=0}^{N-1} H_{s_i s_{i+1}} \qquad H_{s_i s_j} = -\varepsilon s_i s_j - \frac{B}{2}(s_i + s_j)$$

$$Z = \sum_{\{s_i\}} \underbrace{e^{-\beta H}}_{\prod_{i=0}^{N-1} \underbrace{e^{-\beta H_{s_i s_{i+1}}}}_{2\times 2 \text{ matrix } T_{s_i s_{i+1}}}}$$

$$= \sum_{s_0=\pm1}\sum_{s_1=\pm1}\cdots\sum_{s_{N-1}=\pm1} T_{s_0 s_1} T_{s_1 s_2} \cdots T_{s_{N-2} s_{N-1}} T_{s_{N-1} s_0}$$

$$\sum_{s_1} T_{s_0 s_1} T_{s_1 s_2} = (T^2)_{s_0 s_2}$$

$$(AB)_{ij} = \sum_k A_{ik} B_{kj}$$

$$Z = \mathrm{Tr}\, T^N$$

$$T_{ss'} \equiv e^{-\beta H_{ss'}} = \begin{pmatrix} e^{\beta(\varepsilon+B)} & e^{-\beta\varepsilon} \\ e^{-\beta\varepsilon} & e^{\beta(\varepsilon-B)} \end{pmatrix}$$

Interested in $N \to \infty$.

Say T's eigenvalues are λ_0, λ_1 with $\lambda_0 \geq \lambda_1$

Then $\mathrm{Tr}\, T^N = \lambda_0^N + \lambda_1^N$

Ex.: show that

$$\lambda_0 = e^{\beta\varepsilon}\left[\cosh\beta B + \sqrt{\cosh^2\beta B - \left(1 - e^{-4\beta\varepsilon}\right)}\right]$$

As $N \to \infty$,

$$\lim_{N\to\infty} \frac{1}{N} \ln \mathrm{Tr}\, T^N = \mathcal{L}\, \frac{1}{N} \ln\left[\lambda_0^N + \lambda_1^N\right] = \ln \lambda_0$$

Free energy: $f = -\frac{1}{\beta} \ln Z \to$ eq $\approx$ 100.

High T expansion & loop sums

$$Z = \sum_{\{S_i\}} e^{-\beta H(\{S_i\})} = \sum_{\{S_i\}} \sum_{m=0}^{\infty} \frac{[-\beta H]^m}{m!}$$

$$= \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} \sum_{\{S_i\}} [\beta H]^m$$

Consider Ising but zero field

$$Z = \sum_{\{S_i\}} \prod_{\langle i,j \rangle} e^{\beta \varepsilon S_i S_j}$$

i, j nearest neighbors

Simple identity

$$e^{\pm A} = \cosh A \pm \sinh A$$

$$= \cosh A \, [1 \pm \tanh A]$$

$S_i S_j$ always takes values ± 1

so

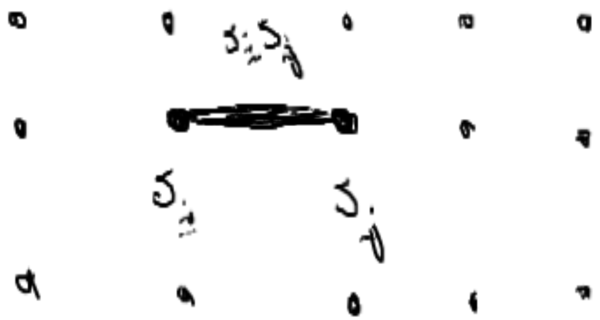
$$Z = \sum_{\{S_i\}} \prod_{\langle i,j \rangle} \cosh(\beta\varepsilon) \, [1 + S_i S_j \tanh \beta\varepsilon]$$

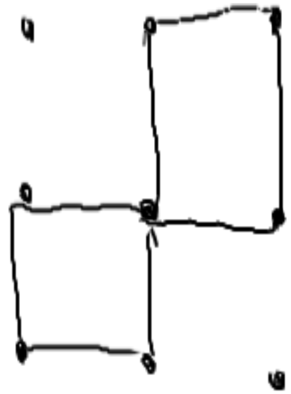
Notation: $\tanh \beta\varepsilon \equiv v$

$$Z = [\cosh \beta\varepsilon]^{Nz/2} \sum_{\{S_i\}} \prod_{\langle i,j \rangle} [1 + S_i S_j v]$$

$$\prod_{\langle i,j\rangle}\left[1+s_i s_j v\right] = 1 + v\sum_{\langle i,j\rangle} s_i s_j + v^2 \sum_{\langle i,j\rangle \neq \langle k,l\rangle} s_i s_j s_k s_l + \dots$$

Graphical repres^{n}:





$$Z = \left[\cosh \beta J\right]^{Nz/2} 2^N \sum_{l=0}^{\infty} g(l)\, v^l$$

$g(l) \equiv$ number of loops of length l on this lattice.

(by convention $g(0) \equiv 1$).

$$\sum_{s_i=\pm1} s_i\left[\dots\right] = 0$$

Idea:

part^{n} f^{n} Z given by sum over closed loops.

Specializing to 2d square lattice

Ising in zero field.

$\overbrace{L \times L}^{N}$ lattice with PBC

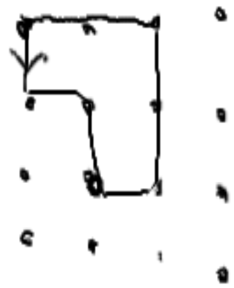
Terminology:

a <u>closed path</u>: an oriented path going from one site to same site

a <u>loop</u>: a pattern of links, each site being visited an even # of times

a path is <u>connected</u> if it's a single body of links

otherwise it's <u>disconnected</u>



Define $D(l) \equiv \frac{1}{2l} \underbrace{h(l)}_{\text{\# of closed connected paths}}$

Guess:

$$g(l) \overset{?}{=} \sum_{m=1}^{l} \frac{1}{m!} \sum_{\substack{l_1, l_2, \dots, l_m \\ l_1+l_2+\dots+l_m = l}} D(l_1) \cdots D(l_m)$$

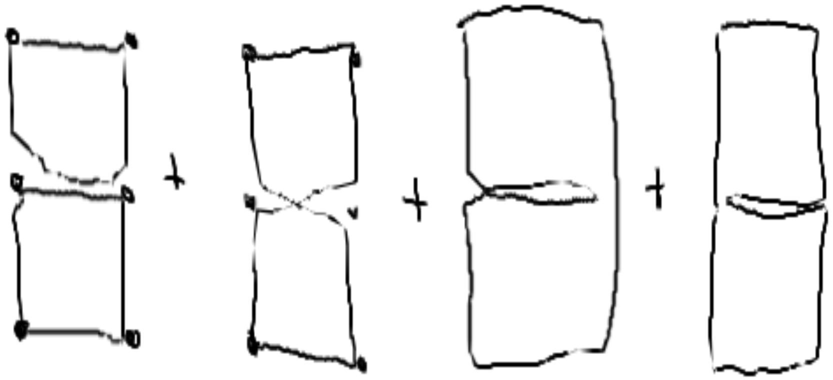
Problem: overcounting

Change rules of game:

turning left: factor $e^{+i\pi/4}$

right: $e^{-i\pi/4}$

Other case:



Introduce matrix $\underset{\sim}{M}$ with elements $M_{\underset{\sim}{i}\underset{\sim}{j}}$

$M_{\underset{\sim}{i}\underset{\sim}{j}}$ is $\neq 0$ only if $\underset{\sim}{i}$ & $\underset{\sim}{j}$ are connected by $\underline{1}$ segment

$$(M^l)_{\underset{\sim}{i}\underset{\sim}{j}}$$

To take phases into account: view each element $M_{\underset{\sim}{i}\underset{\sim}{j}}$ as 4×4 matrix $m^{\alpha\beta}_{\underset{\sim}{i}\underset{\sim}{j}}$

α, β: 0, 1, 2, 3

Convention:

0	E
1	N
2	W
3	S

really, $m^{\alpha\beta}_{\underset{\sim}{i}-\underset{\sim}{j}}$

Explicit example: $m_{(1,0)}$

$$m_{(1,0)} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & e^{i\pi/4} & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & e^{-i\pi/4} & 0 \end{pmatrix}$$

$$m_{(1,0)} = i - j$$

$$m_{(0,1)} = \begin{pmatrix} 0 & 0 & 0 & e^{-i\pi/4} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & e^{i\pi/4} \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Diagonal elements $(M^l)^{\alpha\alpha}_{ii}$

$$D(l) = \frac{-1}{2l} \underset{\substack{\text{over } N\times 4 \\ \text{indices}}}{\operatorname{Tr} M^l} = \frac{-1}{2l} \sum_{i} \operatorname{Tr}_4 (M^l)_{ii}$$

$$Z = [\operatorname{ch}\beta\varepsilon]^{\frac{Nz}{2}} 2^N \sum_{l=0}^{\infty} g(l) v^l$$

$$= [\operatorname{ch}\beta\varepsilon]^{\frac{Nz}{2}} 2^N \left\{ 1 + \sum_{l=1}^{\infty} v^l \sum_{m=1}^{l} \frac{1}{m!} \sum_{\substack{l_1, l_2, \dots \\ l_1 + l_2 + \dots = l}} D(l_1) \dots D(l_m) \right\}$$

$$= [\operatorname{ch}\beta\varepsilon]^{\frac{Nz}{2}} 2^N \left\{ 1 + \sum_{m=1}^{\infty} \frac{1}{m!} \left[\sum_{l=1}^{\infty} D(l) v^l \right]^m \right\}$$

$$Z = [\operatorname{ch}\beta\varepsilon]^{\frac{Nz}{2}} 2^N \exp \sum_{l=1}^{\infty} D(l) v^l$$

$$D(l) = \frac{-1}{2l} \sum_{i} \mathrm{Tr}_4 (M^l)_{ii} = \frac{-1}{2l} \sum_{\alpha=0}^{4N-1} \lambda_\alpha^l$$

Assuming we know the λ_α's, then

$$Z = [\mathrm{ch}\beta\varepsilon]^{\frac{N2}{2}} 2^N \exp\left\{ -\sum_{\alpha=0}^{4N-1} \sum_{l=1}^{\infty} \frac{\lambda_\alpha^l v^l}{2l} \right\}$$

Recognizing exp^n of log: $\ln(1-x) = -\sum_{l=1}^{\infty} \frac{x^l}{l}$

$$Z = [\mathrm{ch}\beta\varepsilon]^{\frac{N2}{2}} 2^N \prod_\alpha (1-\lambda_\alpha v)^{1/2}$$

For the calculation of the eigenvalues, see eq^{n}s 117-120 in notes

Famous expression: eq^n 121

Suggested ex.: eq^n 128, critical temperature of 2d Ising