Statistical Physics & Condensed Matter Theory I: Exercise

BCS ground state: mean and variance of particle number

We have seen that conventional superconductivity is described by the mean-field Hamiltonian (Bogoliubov-Gor'kov-de Gennes; the pairing Δ is chosen to be real using an appropriate gauge)

$$\hat{H} - \mu\hat{N} \simeq \sum_{\mathbf{k}} \left[\xi_{\mathbf{k}} c^{\dagger}_{\mathbf{k}\sigma} c_{\mathbf{k}\sigma} - \Delta(c_{-\mathbf{k}\downarrow} c_{\mathbf{k}\uparrow} + c^{\dagger}_{\mathbf{k}\uparrow} c^{\dagger}_{-\mathbf{k}\downarrow})\right] + \frac{L^d}{g}\Delta^2$$

(in which the fermionic operators obey canonical anticommutation relations $\{c_{\mathbf{k}\sigma}, c^{\dagger}_{\mathbf{k}'\sigma'}\} = \delta_{\mathbf{k}\mathbf{k}'}\delta_{\sigma\sigma'}$) whose ground state is built out of Cooper pairs created on the vacuum according to

$$|\Omega_s\rangle = \prod_{\mathbf{k}} (\cos\theta_{\mathbf{k}} - \sin\theta_{\mathbf{k}} c^{\dagger}_{\mathbf{k}\uparrow} c^{\dagger}_{-\mathbf{k}\downarrow})|0\rangle$$

with parameters (remember that $\xi_{\mathbf{k}} = \varepsilon_{\mathbf{k}} - \mu$)

$$\cos^2\theta_{\mathbf{k}} = \frac{1}{2}\left(1 + \frac{\xi_{\mathbf{k}}}{\sqrt{\Delta^2 + \xi_{\mathbf{k}}^2}}\right), \qquad \sin^2\theta_{\mathbf{k}} = \frac{1}{2}\left(1 - \frac{\xi_{\mathbf{k}}}{\sqrt{\Delta^2 + \xi_{\mathbf{k}}^2}}\right).$$

Very importantly, since the BGdG Hamiltonian above does not conserve particle number, any of its eigenstates will have to combine states with differing fermion numbers. This exercise asks (and answers!) the question: how important are the particle number fluctuations in the BCS ground state?

a) Show explicitly that the BCS ground state $|\Omega_s\rangle$ is normalized.

b) The total particle number operator is given by

$$\hat{N} = \sum_{\mathbf{k}\sigma} c^{\dagger}_{\mathbf{k}\sigma} c_{\mathbf{k}\sigma}.$$

Show that the mean particle number in the BCS ground state is given by

$$\langle N\rangle = \langle\Omega_s|\hat{N}|\Omega_s\rangle = 2\sum_{\mathbf{k}} \sin^2\theta_{\mathbf{k}}.$$

Hint: you can use the fact that the ground state contains equal numbers of spin up and down fermions to write $\hat{N} = 2\hat{N}_{\uparrow} = 2\sum_{\mathbf{k}} c^{\dagger}_{\mathbf{k}\uparrow} c_{\mathbf{k}\uparrow}$ *as far as any expectation value is concerned.*

c) Show that the variance is

$$(\delta N)^2 = \langle\Omega_s|\hat{N}^2|\Omega_s\rangle - \langle\Omega_s|\hat{N}|\Omega_s\rangle^2 = 4\sum_{\mathbf{k}} \cos^2\theta_{\mathbf{k}} \sin^2\theta_{\mathbf{k}}.$$

d)* In view of the explicit form of $\cos\theta_{\mathbf{k}}$ and $\sin\theta_{\mathbf{k}}$ given above, give a rough estimate of the relative standard deviation $\frac{\delta N}{N}$ in terms of the gap Δ, for $\Delta << \varepsilon_F (= \mu)$. Is this standard deviation vanishing in the thermodynamic limit?