Statistical Physics & Condensed Matter Theory I: Exercise

Harmonic oscillator and coherent states

Consider a single harmonic oscillator, in units where $\hbar = 1$, $m = 1$ and $\omega = 1$. The Hamiltonian is thus simply

$$H = a^\dagger a + 1/2.$$

As usual, the bosonic annihilation and creation operators, defined as $a = \frac{1}{\sqrt{2}}(\hat{x} + i\hat{p})$, $a^\dagger = \frac{1}{\sqrt{2}}(\hat{x} - i\hat{p})$, satisfy the canonical commutation relation $[a, a^\dagger] = 1$. We denote the ground state as $|0\rangle$.

a)

Is the ground state $|0\rangle$ a coherent state? Using the $a, a^\dagger$ operators, compute the averages $\bar{x} = \langle 0|\hat{x}|0\rangle$ and $\bar{p} = \langle 0|\hat{p}|0\rangle$. Show that the variances $\Delta x = \sqrt{\langle \hat{x}^2\rangle - \bar{x}^2}$, $\Delta p = \sqrt{\langle \hat{p}^2\rangle - \bar{p}^2}$ saturate (in other words, optimize) Heisenberg's uncertainty relation $\Delta x \Delta p \geq \frac{1}{2}$.

b)

The translation operator $T(\alpha) = e^{-\alpha \frac{\partial}{\partial x}} = e^{-i\alpha\hat{p}}$ shifts the average position of a wavefunction[1]. Show that the shifted ground state $T(\alpha)|0\rangle$ behaves like a coherent state (explicitly: show that it is an eigenstate of the annihilation operator, and give the eigenvalue) [2]. What is this state's normalization? Compute again the averages $\bar{x}, \bar{p}$ and variances. Is Heisenberg's relation still optimized?

c)

Consider now the more general state $|z\rangle = e^{za^\dagger - z^* a}|0\rangle$, in which z is an arbitrary complex number. Is this state normalized? Does it behave like a coherent state? Repeat the calculation of $\bar{x}, \bar{p}$ and variances. Is Heisenberg's relation still optimized? [3]

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[1] This is simply because, in first quantization, we have (using a Taylor series expansion) $\psi(x - \alpha) = \psi(x) - \alpha \frac{\partial}{\partial x}\psi(x) + \frac{\alpha^2}{2}\frac{\partial^2}{\partial x^2}\psi(x) + ... = e^{-\alpha\frac{\partial}{\partial x}}\psi(x)$.

[2] Hint: rewrite everything in terms of $a, a^\dagger$. You will need identities coming from the Campbell-Baker-Hausdorff formula; these are given in the 'Useful formulas' at the end.

[3] Extra info, *not* important for the question: an important set of states, useful in the fields of quantum optics, gravitational waves and many others, are the *squeezed states*. These are given by operating on the vacuum with the squeeze operator $S(z) \equiv e^{\frac{1}{2}z(a^\dagger)^2 - \frac{1}{2}z^* a^2}$. Unlike coherent states, which have $\Delta x = \Delta p$, squeezed states have $\Delta x \neq \Delta p$ (thereby the name), and do not saturate Heisenberg's relation at all times, but only at some specific ones.

Useful formulas

Campbell-Baker-Hausdorff formula

Let A and B be two quantum operators such that $[A, B]$ commutes with A and B. Then, the following identities hold:

$$e^{A+B} = e^A e^B e^{-\frac{1}{2}[A,B]}, \qquad [A, e^{\lambda B}] = \lambda [A, B] e^{\lambda B}.$$

These are a consequence of a more general identity called the Campbell-Baker-Hausdorff formula.