# Coherent states and harmonic oscillator: solution

**a)** Since $a|0\rangle = 0 = 0|0\rangle$, the vacuum is a coherent state of eigenvalue zero. Using $\hat{x} = \frac{1}{\sqrt{2}}(a^\dagger + a)$ and $\hat{p} = \frac{i}{\sqrt{2}}(a^\dagger - a)$, the averages are variances are computed as follows:

$$\begin{aligned}
\bar{x} = \langle 0|\hat{x}|0\rangle &= \frac{1}{\sqrt{2}}\left((\langle 0|a^\dagger)|0\rangle + \langle 0|(a|0\rangle)\right) = 0 + 0 = 0,\\
\bar{p} = \langle 0|\hat{x}|0\rangle &= \frac{i}{\sqrt{2}}\left((\langle 0|a^\dagger)|0\rangle - \langle 0|(a|0\rangle)\right) = 0 - 0 = 0,\\
\Delta x^2 = \langle 0|\hat{x}^2|0\rangle = \frac{1}{2}\langle 0|(a^\dagger + a)^2|0\rangle &= \frac{1}{2}\langle 0|\left((a^\dagger)^2 + a^2 + 2a^\dagger a + 1\right)|0\rangle = \frac{1}{2},\\
\Delta p^2 = \langle 0|\hat{x}^2|0\rangle = \frac{-1}{2}\langle 0|(a^\dagger - a)^2|0\rangle &= \frac{-1}{2}\langle 0|\left((a^\dagger)^2 + a^2 - 2a^\dagger a - 1\right)|0\rangle = \frac{1}{2},
\end{aligned}$$

where we have used the dual relation $\langle 0|a^\dagger = 0$ and normal-ordered the $aa^\dagger$ terms as $a^\dagger a + 1$. We therefore get $\Delta x \Delta p = 1/2$, so we saturate the Heisenberg relation.

**b)** We can write $T(\alpha) = e^{-i\alpha\hat{p}} = e^{\frac{\alpha}{\sqrt{2}}(a^\dagger - a)}$. Using the CBH formula with $A = a$, $B = a^\dagger - a$ and $\lambda = \frac{\alpha}{\sqrt{2}}$, we can write

$$ae^{\frac{\alpha}{\sqrt{2}}(a^\dagger - a)} = \left[a, e^{\frac{\alpha}{\sqrt{2}}(a^\dagger - a)}\right] + e^{\frac{\alpha}{\sqrt{2}}(a^\dagger - a)}a = \frac{\alpha}{\sqrt{2}}e^{\frac{\alpha}{\sqrt{2}}(a^\dagger - a)} + e^{\frac{\alpha}{\sqrt{2}}(a^\dagger - a)}a.$$

Thus, applying this on $|0\rangle$ and using the fact that the vacuum is a coherent state with zero eigenvalue, we get

$$aT(\alpha)|0\rangle = \frac{\alpha}{\sqrt{2}}T(\alpha)|0\rangle + T(\alpha)a|0\rangle = \frac{\alpha}{\sqrt{2}}T(\alpha)|0\rangle$$

so $T(\alpha)|0\rangle$ is a coherent state with eigenvalue $\frac{\alpha}{\sqrt{2}}$. The normalization is given by

$$\langle 0|T^\dagger(\alpha)T(\alpha)|0\rangle = \langle 0|e^{\frac{\alpha}{\sqrt{2}}(a - a^\dagger)}e^{\frac{\alpha}{\sqrt{2}}(a^\dagger - a)}|0\rangle = \langle 0|1|0\rangle = 1$$

where we have used that $T^\dagger T = 1$ as seen from CBH with $A = -B = \frac{\alpha}{\sqrt{2}}(a - a^\dagger)$ ($[A, B]$ is then trivially 0).

The averages and variances are computed as follows, using this last result and its dual $\langle 0|T^\dagger(\alpha)a^\dagger = \frac{\alpha}{\sqrt{2}}\langle 0|T^\dagger(\alpha)$:

$$\bar{x} = \frac{1}{\sqrt{2}}\langle 0|T^\dagger(\alpha)(a^\dagger + a)T(\alpha)|0\rangle = \frac{2}{\sqrt{2}}\frac{\alpha}{\sqrt{2}} = \alpha, \quad \bar{p} = \frac{i}{\sqrt{2}}\langle 0|T^\dagger(\alpha)(a^\dagger - a)T(\alpha)|0\rangle = \frac{(1-1)}{\sqrt{2}}\frac{\alpha}{\sqrt{2}} = 0,$$

$$\Delta x^2 = \langle \hat{x}^2\rangle - \bar{x}^2 = \frac{1}{2}\langle 0|T^\dagger(\alpha)\left((a^\dagger)^2 + a^2 + 2a^\dagger a + 1\right)T(\alpha)|0\rangle - \alpha^2 = \frac{1}{2}(\frac{\alpha^2}{2} + \frac{\alpha^2}{2} + 2\frac{\alpha^2}{2} + 1) - \alpha^2 = \frac{1}{2},$$

$$\Delta p^2 = \frac{-1}{2}\langle 0|T^\dagger(\alpha)\left((a^\dagger)^2 + a^2 - 2a^\dagger a - 1\right)T(\alpha)|0\rangle = \frac{-1}{2}(\frac{\alpha^2}{2} + \frac{\alpha^2}{2} - 2\frac{\alpha^2}{2} - 1) = \frac{1}{2}.$$

so Heisenberg's relation is still optimized.

**c)** Normalization: again, use CBH with $A = -za^\dagger + z^* a = -B$ to get $\langle 0|e^{-z^* a + z^* a^\dagger} e^{za^\dagger - z^* a}|0\rangle = 1$. The state behaves like a coherent state since

$$ae^{za^\dagger - z^* a}|0\rangle = \left([a, e^{za^\dagger - z^* a}] + e^{za^\dagger - z^* a} a\right)|0\rangle = [a, za^\dagger - z^* a]e^{za^\dagger - z^* a}|0\rangle + 0 = ze^{za^\dagger - z^* a}|0\rangle.$$

The averages and variances are thus, using the notation $e^{za^\dagger - z^* a}|0\rangle \equiv |z\rangle$ such that $a|z\rangle = z|z\rangle$, $\langle z|a^\dagger = z^*\langle z|$:

$$\bar{x} = \langle z|\frac{1}{\sqrt{2}}(a^\dagger + a)|z\rangle = \frac{z^* + z}{\sqrt{2}}, \qquad \bar{p} = \langle z|\frac{i}{\sqrt{2}}(a^\dagger - a)|z\rangle = i\frac{z^* - z}{\sqrt{2}},$$
$$\Delta x^2 = \frac{1}{2}\langle z|\left((a^\dagger)^2 + a^2 + 2a^\dagger a + 1\right)|z\rangle - \bar{x}^2 = \frac{1}{2}\left((z^*)^2 + z^2 + 2|z|^2 + 1\right) - \frac{(z^* + z)^2}{2} = \frac{1}{2},$$
$$\Delta p^2 = \frac{-1}{2}\langle z|\left((a^\dagger)^2 + a^2 - 2a^\dagger a - 1\right)|z\rangle - \bar{p}^2 = \frac{-1}{2}\left((z^*)^2 + z^2 - 2|z|^2 - 1\right) + \frac{(z^* - z)^2}{2} = \frac{1}{2}$$

so Heisenberg's relation is still optimized.