Statistical Physics & Condensed Matter Theory I: Exercise

The resonant level model: Solution

a)

The total number of fermions will be conserved under time evolution provided $[H_{RLM}, N_f] = 0$. We can compute this term-by-term. Besides the obviously vanishing terms $[c_k^\dagger c_k, c_{k'}^\dagger c_{k'}] = 0$, $[c_k^\dagger c_k, d^\dagger d] = 0$, $[d^\dagger d, d^\dagger d] = 0$, we have the nontrivial commutators

$$\begin{aligned}
\left[c_k^\dagger d, c_{k'}^\dagger c_{k'}\right] &= c_k^\dagger d c_{k'}^\dagger c_{k'} - c_{k'}^\dagger c_{k'} c_k^\dagger d = d c_{k'}^\dagger \{c_k^\dagger, c_{k'}\} = -\delta_{k,k'} c_k^\dagger d, \\
\left[d^\dagger c_k, c_{k'}^\dagger c_{k'}\right] &= d^\dagger c_k c_{k'}^\dagger c_{k'} - c_{k'}^\dagger c_{k'} d^\dagger c_k = d^\dagger \{c_k, c_{k'}^\dagger\} c_{k'} = \delta_{k,k'} d^\dagger c_k,
\end{aligned}$$

and (remembering that $d^2 = 0$ and $(d^\dagger)^2 = 0$),

$$\begin{aligned}
\left[c_k^\dagger d, d^\dagger d\right] &= c_k^\dagger d d^\dagger d - d^\dagger d c_k^\dagger d = c_k^\dagger (1 - d^\dagger d) d - 0 = c_k^\dagger d, \\
\left[d^\dagger c_k, d^\dagger d\right] &= d^\dagger c_k d^\dagger d - d^\dagger d d^\dagger c_k = 0 - d^\dagger (1 - d^\dagger d) c_k = -d^\dagger c_k.
\end{aligned}$$

Therefore,

$$\begin{aligned}
[H_{RLM}, N_f] &= t \sum_{k,k'} \left[c_k^\dagger d + d^\dagger c_k, c_{k'}^\dagger c_{k'}\right] + t \sum_k \left[c_k^\dagger d + d^\dagger c_k, d^\dagger d\right] \\
&= t \sum_k \left(-c_k^\dagger d + d^\dagger c_k + c_k^\dagger d - d^\dagger c_k\right) = 0.
\end{aligned}$$

b)

We need to explicitly calculate (dropping obviously zero terms proportional to $[c_k^\dagger c_k, d^\dagger] = 0$ and $[d^\dagger d, c_k] = 0$ and the like)

$$\left[H_{RLM}, f_n^\dagger\right] = \sum_{k,k'} \varepsilon_k M_{n,k'} \left[c_k^\dagger c_k, c_{k'}^\dagger\right] + L_n \varepsilon_d \left[d^\dagger d, d^\dagger\right] + t \left(L_n \sum_k \left[c_k^\dagger d, d^\dagger\right] + \sum_{k,k'} M_{n,k'} \left[d^\dagger c_k, c_{k'}^\dagger\right] \right)$$

$$= \sum_k \varepsilon_k M_{n,k} c_k^\dagger + L_n \varepsilon_d d^\dagger + t L_n \sum_k c_k^\dagger + t \sum_k M_{n,k} d^\dagger = \sum_k (\varepsilon_k M_{n,k} + t L_n) c_k^\dagger + (\varepsilon_d L_n + t \sum_k M_{n,k}) d^\dagger$$

(using *e.g.* $[c_k^\dagger d, d^\dagger] = c_k^\dagger d d^\dagger - d^\dagger c_k^\dagger d = c_k^\dagger \{d, d^\dagger\} = c_k^\dagger$ and similar). Equating this to $E_n f_n^\dagger$ yields the conditions

$$E_n M_{n,k} = \varepsilon_k M_{n,k} + t L_n, \qquad E_n L_n = \varepsilon_d L_n + t \sum_k M_{n,k}$$

which are the sought-after coupled equations for $M_{n,k}$ and L_n.

c)

Using $M_{n,k} = tL_n/(E_n - \varepsilon_k)$, we get $(E_n - \varepsilon_d)L_n = t^2 L_n \sum_k \frac{1}{E_n - \varepsilon_k}$ and therefore

$$E_n - \varepsilon_d = \sum_k \frac{t^2}{E_n - \varepsilon_k}.$$

d)

We have

$$\begin{aligned}\langle 0|f_n f_n^\dagger|0\rangle &= \langle 0|(\sum_k (M_{n,k})^* c_k + L_n^* d)(\sum_{k'} M_{n,k'} c_{k'}^\dagger + L_n d)|0\rangle \\ &= \sum_{k,k'} (M_{n,k})^* M_{n,k'} \langle 0|c_k c_{k'}^\dagger|0\rangle + |L_n|^2 \langle 0|dd^\dagger|0\rangle = \sum_k |M_{n,k}|^2 + |L_n|^2\end{aligned}$$

and thus (substituting for $M_{n,k}$ from above)

$$|L_n| = \left(1 + \sum_k \frac{t^2}{(E_n - \varepsilon_k)^2}\right)^{-1/2}.$$

e)

For your enlightenment, let's first prove the identity given in the question:

$$\begin{aligned}\sum_{k\in\mathbb{Z}} \frac{1}{E_n - \varepsilon_k} &= \sum_{k\in\mathbb{Z}} \frac{1}{E_n - \Delta(k - 1/2)} = \frac{2}{\Delta} \sum_{k\in\mathbb{Z}} \frac{1}{\frac{2E_n}{\Delta} - (2k-1)} \\ &= \frac{2}{\Delta} \sum_{k=1}^{\infty} \left(\frac{1}{\frac{2E_n}{\Delta} - (2k-1)} + \frac{1}{\frac{2E_n}{\Delta} + (2k-1)} \right) = \frac{2}{\Delta} \sum_{k=1}^{\infty} \frac{4E_n/\Delta}{(\frac{2E_n}{\Delta})^2 - (2k-1)^2} = -\frac{\pi}{\Delta} \tan\left(\frac{\pi E_n}{\Delta}\right).\end{aligned}$$

Thus,

$$\sum_{k\in\mathbb{Z}} \frac{1}{(E_n - \varepsilon_k)^2} = -\frac{\partial}{\partial E_n}\left(-\frac{\pi}{\Delta}\tan\left(\frac{\pi E_n}{\Delta}\right)\right) = \frac{\pi^2}{\Delta^2} \frac{1}{\cos^2 \frac{\pi E_n}{\Delta}}$$

But we also have

$$\frac{E_n - \varepsilon_d}{t^2} = \sum_k \frac{1}{E_n - \varepsilon_k} = -\frac{\pi}{\Delta} \tan\left(\frac{\pi E_n}{\Delta}\right)$$

so

$$\frac{(E_n - \varepsilon_d)^2}{t^4} = \frac{\pi^2}{\Delta^2}\left(\frac{1}{\cos^2 \frac{\pi E_n}{\Delta}} - 1\right) \to \frac{1}{\cos^2 \frac{\pi E_n}{\Delta}} = 1 + \frac{\Delta^2}{\pi^2 t^4}(E_n - \varepsilon_d)^2.$$

This allows us to write

$$|L_n|^{-2} = 1 + \frac{\pi^2 t^2}{\Delta^2} + \left(\frac{E_n - \varepsilon_d}{t}\right)^2$$

so finally

$$|L_n|^2 = \frac{t^2}{(E_n - \varepsilon_d)^2 + t^2(1 + \pi^2 t^2/\Delta^2)},$$

and this has Lorentzian form $I/((E_n - \alpha)^2 + \gamma^2)$ with $I = t^2$, $\alpha = \varepsilon_d$ and inverse lifetime $\gamma = t(1 + \pi^2 t^2/\Delta^2)^{1/2}$.