Statistical Physics & Condensed Matter Theory I: Exercise

Schwinger bosons

Consider two bosonic operators a and b, obeying the standard canonical bosonic operator commutation relations $[a, a^\dagger] = 1$, $[b, b^\dagger] = 1$ (all other commutators are zero).

a) Show that one can use the **Schwinger boson representation** to write spin operators as

$$S^+ = a^\dagger b, \qquad S^- = b^\dagger a, \qquad S^z = \frac{1}{2}(n_a - n_b),$$

(in which $n_a \equiv a^\dagger a$ and $n_b \equiv b^\dagger b$) by explicitly verifying (using the bosonic commutation relations) that these spin operators obey the $su(2)$ algebra

$$\left[S^+, S^-\right] = 2S^z, \qquad \left[S^z, S^\pm\right] = \pm S^\pm.$$

b) Again using the bosonic commutation relations, show that the state

$$|S, m\rangle \equiv \frac{(a^\dagger)^{S+m}}{\sqrt{(S+m)!}} \frac{(b^\dagger)^{S-m}}{\sqrt{(S-m)!}} |\Omega\rangle$$

(in which $|\Omega\rangle$ is the vacuum of the a and b operators, *i.e.* $a|\Omega\rangle = 0$, $b|\Omega\rangle = 0$) is an eigenstate of the total spin $\mathbf{S}^2$ and S^z operators with the correct eigenvalues ($S(S+1)$ and m respectively, where we take $n_a + n_b = 2S$).