Lecture 10a

Mean field theory

Hubbard model

$$\hat{H} = -t \sum_{\substack{\langle ij \rangle \\ \sigma}} \hat{c}^\dagger_{i\sigma} \hat{c}_{j\sigma} + U \sum_i \hat{n}_{i\uparrow} \hat{n}_{i\downarrow}$$

"vacuum exp. value (vev)"

$$\hat{n}_{i,\sigma} = \hat{c}^\dagger_{i\sigma} \hat{c}_{i\sigma}$$

Order parameter

$$\langle \hat{S}^z_i \rangle = \frac{1}{2} \left(\langle \hat{n}_{i\uparrow} \rangle - \langle \hat{n}_{i\downarrow} \rangle \right)$$

$$\neq 0.$$

$$\hat{n}_{i\uparrow} = \langle n_{i\uparrow} \rangle + \widehat{\delta n}_{i\uparrow}$$

$\leftarrow$ quantum fluctuations

$$\hat{H}_{int} = U \sum_i \hat{n}_{i\uparrow} \hat{n}_{i\downarrow} = U \sum_i \langle n_{i\uparrow} \rangle \langle n_{i\downarrow} \rangle + \widehat{\delta n}_{i\uparrow} \langle n_{i\downarrow} \rangle$$

$$+ \langle n_{i\uparrow} \rangle \widehat{\delta n}_{i\downarrow} + \cancel{\widehat{\delta n}_{i\uparrow} \widehat{\delta n}_{i\downarrow}}$$

$$\widehat{\delta n}_{i\uparrow} = \hat{n}_{i\uparrow} - \langle n_{i\uparrow} \rangle$$

$$\hat{H} = -t \sum_{\langle ij \rangle \sigma} \hat{c}^{+}_{i\sigma} \hat{c}_{j\sigma} + U \sum_i \langle n_{i\uparrow} \rangle \hat{n}_{i\downarrow} + \langle n_{i\downarrow} \rangle \hat{n}_{i\uparrow}$$

$$- \langle n_{i\uparrow} \rangle \langle n_{i\downarrow} \rangle$$

$$\longrightarrow \underbrace{\hat{c}^\dagger_{i\uparrow}\,\hat{c}_{i'\uparrow}}\;\underbrace{\hat{c}^\dagger_{i\downarrow}\,\hat{c}_{i\downarrow}}$$

$$\longrightarrow \underbrace{\hat{c}^\dagger_{i\uparrow}\,\hat{c}^\dagger_{i\downarrow}}\;\underbrace{\hat{c}_{i'\downarrow}\,\hat{c}_{i'\uparrow}}$$

$$\sigma_i \equiv \frac{1}{2}\left(\langle n_{i\uparrow}\rangle - \langle n_{i\downarrow}\rangle\right) \qquad \langle n_{i\uparrow}\rangle = \rho_i + \sigma_i$$

$$\rho_i \equiv \frac{1}{2}\left(\langle n_{i\uparrow}\rangle + \langle n_{i\downarrow}\rangle\right) \qquad \langle n_{i\downarrow}\rangle = \rho_i - \sigma_i$$

$$\hat{H}^{MF} = -t \sum_{\langle ij\rangle, s} \hat{c}^\dagger_{i,s}\hat{c}_{j,s} + U \sum_i \Big\{ \rho_i \left(\hat{c}^\dagger_{i\uparrow}\hat{c}_{i\uparrow} + \hat{c}^\dagger_{i\downarrow}\hat{c}_{i\downarrow}\right) - \sigma_i \left(\hat{c}^\dagger_{i\uparrow}\hat{c}_{i\uparrow} - \hat{c}^\dagger_{i\downarrow}\hat{c}_{i\downarrow}\right) \underline{\underline{- \rho_i^2 + \sigma_i^2}} \Big\}$$

$\rho_i \to \rho$

$\sigma_i \to (-1)^i \sigma$

"Ansatz"

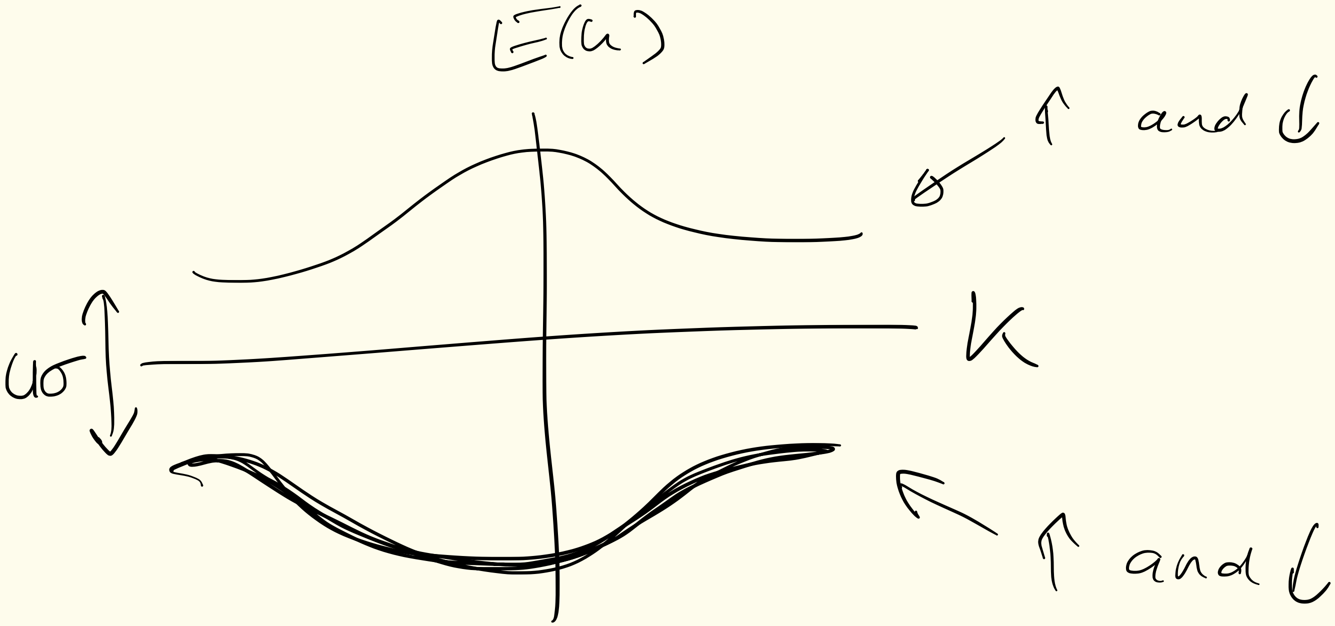
$\hat{c}^\dagger_i \to \hat{a}^\dagger_i \qquad i$ even

$\quad\;\; \to \hat{b}^\dagger_i \qquad i$ odd

$$\hat{H}^{MF} = \sum_k \left(\hat{a}^\dagger_{k\uparrow} \; \hat{b}^\dagger_{k\uparrow} \; \hat{a}^\dagger_{k\downarrow} \; \hat{b}^\dagger_{k\downarrow} \right) \left(\begin{array}{cc|cc} U(\rho-\sigma) & -2t\cos(\frac{ka}{2}) & 0 & 0 \\ -2t\cos(\frac{ka}{2}) & U(\rho+\sigma) & 0 & 0 \\ \hline 0 & 0 & U(\rho+\sigma) & -2t\cos() \\ 0 & 0 & -2t\cos() & U(\rho-\sigma) \end{array} \right) \begin{pmatrix} \hat{a}_{k\uparrow} \\ \hat{b}_{k\uparrow} \\ \hat{a}_{k\downarrow} \\ \hat{b}_{k\downarrow} \end{pmatrix}$$

$$H = \sum_k \chi^\dagger_k E_k \chi_k$$

$$\boxed{E^{\pm}(k) = U(\sigma^2 - \rho^2) + U\rho \pm \sqrt{U^2\sigma^2 + 4t^2\cos^2(\frac{ka}{2})}}$$



$uU\sigma^2$

gain: $u\sigma$

loose: $u\sigma^2$

$$E^{MF} = \frac{2}{N}\sum_k \Big(U(\sigma^2-\rho^2) + U\rho - \sqrt{U^2\sigma^2 + 4t^2\cos^2\left(\frac{ka}{2}\right)} \Big)$$

$$\frac{\partial E}{\partial \rho} = \frac{\partial E}{\partial \sigma} = 0$$ "Saddle point equations"

$$\begin{cases} -2\rho U + \frac{2}{N}\sum_k \frac{\partial}{\partial \rho}\Big(U\rho - \sqrt{U^2\sigma^2 + 4t^2\cos^2\left(\frac{ka}{2}\right)}\Big) = 0 \\ 2\sigma U + \frac{2}{N}\sum_k \frac{\partial}{\partial \sigma} \quad '' \quad = 0 \end{cases}$$

$$-2\rho \underline{u} + \frac{2}{N}\sum_k \underline{u} = 0$$

$$\sum_k 1 = \frac{N}{2}$$

$$\Rightarrow \rho = 1/2$$

$c_i^\dagger \quad i = 1 \ldots N$

$c_k^\dagger \quad k = -\frac{\pi}{a} \ldots \frac{\pi}{a}$

steps $\frac{2\pi}{Na}$

$a_n^\dagger = \quad n = 1, 2, 3 \ldots N/2$

$a_k^\dagger \quad k = -\frac{\pi}{2a} \ldots \frac{\pi}{2a}$

$a_i^\dagger \quad i = 2, 4, 6 \ldots N$

$b_i^\dagger \quad i = 1, 3, 5, \ldots N-1$

$$2\sigma u + \frac{2}{N}\sum_k \frac{-2u^2\sigma}{2\sqrt{u^2\sigma^2 + 4t^2\cos^2\left(\frac{ka}{2}\right)}} = 0$$

$$\boxed{\frac{1}{4\pi}\int_{-\pi}^{\pi} dk \,\frac{1}{\sqrt{1+\left(\frac{2t}{u\sigma}\right)^2\cos^2\left(\frac{ka}{2}\right)}} = \sigma}$$

Self consistent solution.

$$\sigma \approx \frac{1}{4\pi}\int_{-\pi}^{\pi} dk \left[1 - 2\left(\frac{t}{u\sigma}\right)^2 \cos^2\left(\frac{ka}{2}\right)\right]$$

$U/t \gg 1$

$$= \frac{1}{2}\left(1 - \left(\frac{t}{u\sigma}\right)^2\right)$$

$$\hat{\alpha}^{+}_{i,\uparrow} \propto \hat{a}^{+}_{i,\uparrow} - \frac{t}{u}\left(\hat{b}^{+}_{i+1,\uparrow} + \hat{b}^{+}_{i-1,\uparrow}\right)$$ ← occupied

$\hat{\beta}^{+}_{i,\uparrow}$ - - -

$$\hat{\alpha}^{+}_{i,\downarrow} \propto \hat{a}^{+}_{i,\downarrow} + \frac{t}{u}\left(\hat{b}^{+}_{i+1,\downarrow} + \hat{b}^{+}_{i-1,\downarrow}\right)$$ ← empty

$\hat{\beta}^{+}_{i,\downarrow}$ - - -

$U \ll t$: Spin Density Wave

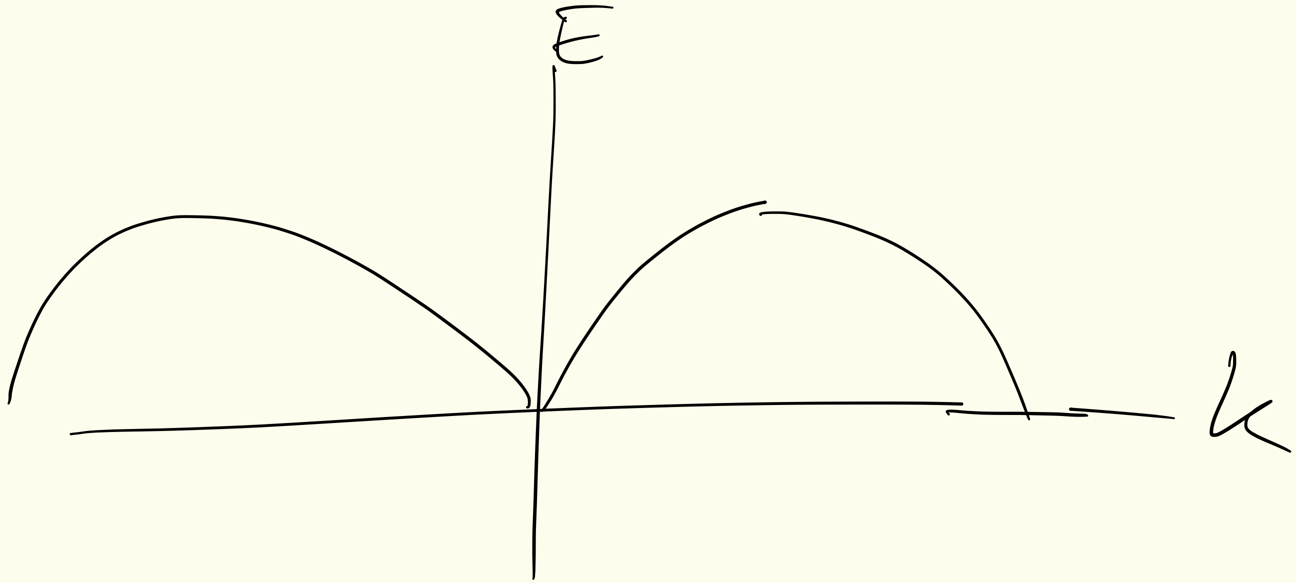
SDW $\quad \sigma \ll 1$

$$\hat S^+_{i\alpha} = \hat\alpha^\dagger_{i\uparrow}\hat\alpha_{i\downarrow}$$

$$\cancel{\hat a^\dagger_{i\uparrow}\hat a_{i\downarrow}}$$

Magnons

$$i\hbar\,\dot{\hat S}^+ = [\hat S^+, \hat H^{MF}]$$



$$\widehat{\delta n_{i\uparrow}} = \hat{n}_{i\uparrow} - \langle n_{i\uparrow} \rangle$$

$\alpha^{\dagger}\beta \ldots$

$\rho + \sigma$

Lecture 10b

weakly interacting Fermi gas

$$S[\bar{\psi},\psi] = S_0 + S_{int}$$

$$S_0 = \sum_{p\sigma} \bar{\psi}_{p,\sigma}\left(-i\omega_n + \frac{|\vec{p}|^2}{2m} - \mu\right)\psi_{p,\sigma}$$

$$S_{int} = \frac{T}{2L^3}\sum_q \rho_q V(q)\rho_{-q} \qquad \rho_q \equiv \sum_{p\sigma} \bar{\psi}_{p+q,\sigma}\psi_{p,\sigma}$$

$$\int \underline{\underline{D\phi}}\; e^{-\frac{e^2\beta}{2L^3}\sum_q \phi_q V^{-1}_{(q)} \phi_{-q}} \equiv 1$$

ϕ = boson

$$\phi_q \to \phi_q - \frac{i}{e\beta} V(q)\, \rho_q$$

$$1 = \int D\phi\; e^{\sum_q \left(-\frac{e^2\beta}{2L^3} \phi_q V^{-1}_{(q)} \phi_{-q} + \frac{ie}{2L^3}\left(\phi_q \rho_{-q} + \phi_{-q}\rho_q\right) + \underset{\substack{\uparrow \\ S_{int}}}{\frac{1}{2\beta L^3} \rho_q V(q) \rho_{-q}} \right)}$$

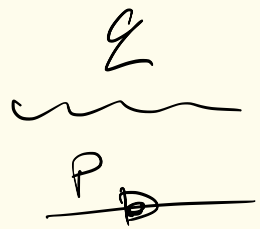
$$\Rightarrow e^{-S_{int}} = \int D\phi\; e^{\sum_q \left(-\frac{e^2\beta}{2L^3} \phi_q V^{-1}_{(q)} \phi_{-q} + \frac{ie}{L^3} \phi_q \rho_{-q} \right)}$$

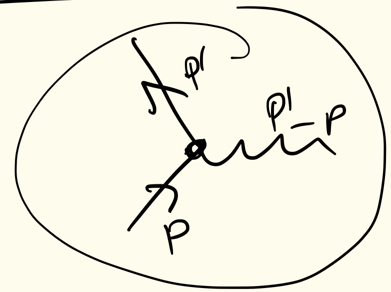
$$Z = \int D\phi \int D[\bar{\psi},\psi]\, e^{-S[\phi,\bar{\psi},\psi]}$$

$$\boxed{S = \frac{\beta}{8\pi L^3}\sum_q \phi_q\, q^2\, \phi_{-q} + \sum_{p,\sigma} \bar{\psi}_{p,\sigma}\left(-i\omega_n + \frac{|\vec{p}|^2}{2m} - \mu\right)\psi_{p,\sigma} + \frac{ie}{L^3}\sum_{pp'\sigma} \bar{\psi}_{p\sigma}\,\psi_{p'\sigma}\,\phi_{p'-p}}$$

$$V(q) = \frac{4\pi e^2}{q^2}$$

Decoupled interaction term.





$$\boxed{\text{Hubbard-Stratonovich transformation}}$$

$\cancel{\bar{\psi}\psi\phi}$

$$Z = \int D\phi \; e^{-\frac{\beta}{8\pi L^3} \sum_q \phi_q q^2 \phi_{-q}} \cdot \det\left[-i\hat{\omega} + \frac{\hat{p}^2}{2m} - \mu + \frac{ie}{L^3}\hat{\phi}\right]$$

$$\sum_{p p'} \bar{\psi}_p \psi_{p'} \phi_{p'-p} = \underline{\vec{\psi}^T \hat{\phi} \vec{\psi}}$$

$\vec{\psi}$ components ψ_p

$(\hat{\omega})_{pp'} = \omega_n \delta_{p,p'}$

$\hat{P}_{pp'} = \vec{P} \delta_{p,p'}$

$\hat{\phi}$ matrix elements

$$(\hat{\phi})_{p,p'} = \phi_{p'-p}$$

$$\boxed{\ln \det \hat{A} = \mathrm{Tr} \ln \hat{A}}$$

$$Z = \int D\phi \; e^{-S[\phi]}$$

$$S = \frac{\beta}{8\pi L^3} \sum_q \underline{\phi_q q^2 \phi_{-q}} - \mathrm{Tr} \ln \underline{\left[-i\hat{\omega} + \frac{\hat{p}^2}{2m} - \mu + \frac{ie}{L^3}\hat{\phi} \right]}$$

$$\boxed{\frac{\delta S}{\delta \phi_q} = 0}$$

Sds =

<u>Mean field sds</u>

Lecture 10c

$$S[\phi] = \frac{\beta}{8\pi L^3} \sum_q \phi_q q^2 \phi_{-q} - \mathrm{Tr}\,\mathrm{Ln}\left[-i\hat{\omega} + \frac{\hat{p}^2}{2m} - \mu + \frac{ie}{L^3}\hat{\phi}\right]$$

$$\frac{\delta S[\phi]}{\delta \phi_q} = 0$$

$$\boxed{\frac{\partial}{\partial x} \mathrm{Tr}\left(f(\hat{A})\right) = \mathrm{Tr}\left(f'(\hat{A}) \frac{\partial}{\partial x} \hat{A}\right)}$$

$$\hat{G}^{-1} \equiv -i\hat{\omega} + \frac{\hat{p}^2}{2m} - \mu + \frac{ie}{L^3}\hat{\phi}$$

$$\mathrm{Tr}\,\hat{C} = \sum_i C_{ii}$$

$$\hat{C} = \hat{A}\hat{B}$$

$$C_{ij} = \sum_m A_{im} B_{mj}$$

$$\frac{\delta}{\delta \phi_q} \mathrm{Tr}\,\mathrm{Ln}\,\hat{G}^{-1} = \mathrm{Tr}\left(\hat{G} \frac{\delta}{\delta \phi_q} \hat{G}^{-1}\right)$$

$$= \sum_{pp'} (\hat{G})_{p,p'} \left(\frac{\delta}{\delta \phi_q} \hat{G}^{-1}\right)_{p',p}$$

$$\hat{g}$$

$$\hat{\phi}$$

$$\hat{\phi}_{p,p'} = \phi_{p'-p}$$

$$\left(\frac{\delta}{\delta\phi_q}\hat{\phi}\right)_{p,p'} = \frac{\delta}{\delta\phi_q}\phi_{p,p'}$$

$$= \frac{\delta}{\delta\phi_q}\phi_{p'-p}$$

$$= \delta_{q,p'-p}$$

$$\frac{\delta}{\delta \phi_q} S = \frac{\beta}{4\pi L^3} q^2 \phi_{-q} + \frac{2ie}{L^3} \sum_p \left(\hat{g}\right)_{p,p-q} = 0$$

saddle point eq.

$$\phi_{q=0} = 0 \qquad V(q=0) = 0$$

$$\phi_q = 0 \quad \forall q \qquad \hat{\phi}_{p,p-q} = 0 \qquad \hat{p}, \hat{\omega} \propto \delta_{p,p'}$$

$$\left(\hat{g}\right)_{p,p-q} = 0$$

$$q \neq 0$$

$$\hat{g}^{-1} = -i\hat{\omega} + \frac{\hat{p}^2}{2m} - \mu + \frac{ie}{L^3} \hat{\phi}$$

$$\phi_q = \phi_q^{MF} + \delta\phi_q$$

$$= 0 + \delta\phi_q$$

$$\mathrm{Tr}\,\mathrm{Ln}\,\hat{G}^{-1} = \mathrm{Tr}\,\mathrm{Ln}\,\underline{\underline{\hat{G}_0^{-1}}}$$

expand $\hat{G}^{-1}$

for $\underline{\underline{\text{small}}}$ ϕ_q

$$+ \frac{ie}{L^3}\,\underline{\underline{\mathrm{Tr}(\hat{G}_0\hat{\phi})}}$$

$$+ \frac{1}{2}\left(\frac{e}{L^3}\right)^2 \mathrm{Tr}(\hat{G}_0\hat{\phi}\hat{G}_0\hat{\phi}) + \ldots$$

$$\hat{G}_0^{-1} = \underline{i\hat{\omega} + \frac{\hat{p}^2}{2m} - \mu}$$

$$(\hat{G}_0)_{p,p'} = G_0(p)\,\delta_{p,p'}$$

$$e^{\mathrm{Tr}\,\mathrm{Ln}\,G_0^{-1}} = e^{-\mathrm{Tr}\,\mathrm{Ln}\,G_0}$$

$$= \det(G_0^{-1}) = Z_0$$

$$\mathrm{Tr}(\hat{g}_0 \hat{\phi}) = \sum_{p p'} (\hat{g}_0)_{p,p'} (\hat{\phi})_{p' p}$$

$$= \sum_{p p'} g_0(p) \, \delta_{p,p'} \, \phi_{p-p'}$$

$$= \sum_{p} g_0(p) \, \phi_0$$

$$= 0$$

$$\phi_0 = 0$$

$$V(q=0) = 0$$

$$\frac{1}{2}\left(\frac{e}{L^3}\right)^2 \sum_{p_1 p_2 p_3 p_4} (\hat{g}_0)_{p_1 p_2} (\hat{\phi})_{p_2 p_3} (\hat{g}_0)_{p_3 p_4} (\hat{\phi})_{p_4 p_1}$$

$$= \frac{1}{2}\left(\frac{e}{L^3}\right)^2 \sum g_0(p_1)\, \delta_{p_1,p_2}\, \phi_{p_3-p_2}\, g_0(p_3)\, \delta_{p_3,p_4}\, \phi_{p_1-p_4}$$

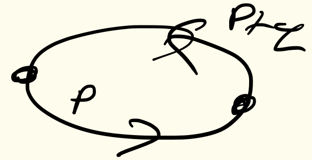
$$= \sum_{p p'} \frac{1}{2}\left(\frac{e}{L^3}\right)^2 g_0(p)\, g_0(p')\, \phi_{p'-p}\, \phi_{p-p'}$$

$$= \sum_{p q} \frac{1}{2}\left(\frac{e}{L^3}\right)^2 \underline{g_0(p)\, g_0(p+q)}\, \phi_q\, \phi_{-q}$$

$$= \underline{\underline{\frac{e^2}{2TL^3} \sum_q \Pi_q\, \phi_q\, \phi_{-q}}}$$

Polarization

$$\Pi_q = \frac{2T}{L^3} \sum_p g_0(p)\, g_0(p+q)$$



$$S^{(2)}_{eff}[\phi] = \frac{1}{2\pi L^3} \sum_q \phi_q \underbrace{(q^2 - e^2 \Pi_q)}_{G^{-1}} \phi_{-q}$$

$$G(q) = \frac{1}{q^2 - e^2 \Pi_q}$$

$$Z_{eff} = \int D\phi \, e^{-S_{eff}}$$

$$= Z_0 \prod_q \left(1 - \frac{4\pi e^2}{q^2} \Pi_q\right)^{-1/2}$$

$$F_{eff} = -T \ln Z_{eff}$$

$$= F_0 + \frac{T}{2} \sum_q \ln\left(1 - V(q) \Pi_q\right)$$

$$= F^{RPA}.$$