6 Effective theories / Mean field theory

$$H = H_0 + H_{int} \qquad H_{int} = O_1 O_2$$

Trick: rewrite $O_i = \langle O_i \rangle_{MF} + \underbrace{\left(O_i - \langle O_i \rangle_{MF}\right)}_{\delta O_i}$

$$H_{int} = \langle O_1 \rangle_{MF} \langle O_2 \rangle_{MF} + \langle O_1 \rangle_{MF} \delta O_2 + \langle O_2 \rangle_{MF} \delta O_1 + \delta O_1 \delta O_2$$

Assume fluctuations to be small w.r.t. to exp^n value, can "drop" the $\delta O \delta O$ 2nd order term.

Replace H by $H_{eff} = H_0 + \langle O_1 \rangle_{MF} O_2 + O_1 \langle O_2 \rangle_{MF} - \langle O_1 \rangle_{MF} \langle O_2 \rangle_{MF}$

H_{eff} is usually exactly solvable (1-body terms only)

Expectation values in effective theory:

$$\langle O_i \rangle_{\text{eff}} = \frac{1}{Z_{\text{eff}}} \mathrm{Tr}\, O_i \, e^{-\beta H_{\text{eff}}} \qquad Z_{\text{eff}} = \mathrm{Tr}\, e^{-\beta H_{\text{eff}}}$$

Must ensure self-consistency:

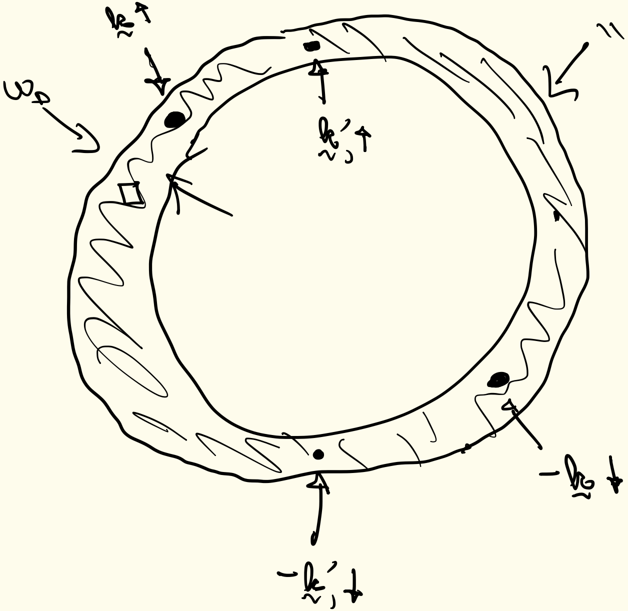
require $\langle O_i \rangle_{\text{eff}} = \langle O_i \rangle_{MF}$

Could compute e.g. $\left\langle \left(O_i - \langle O_i \rangle_{MF} \right)^2 \right\rangle_{\text{eff}}$

$$\left\langle \left(O_i - \langle O_i \rangle_{MF} \right) \right\rangle_{\text{eff}} = \langle O_i \rangle_{\text{eff}} - \langle O_i \rangle_{MF} \equiv 0$$

by self-cons.

BCS superconductivity



"layer" of e^-'s able to interact through phonons (width $\sim \omega_D$)

Int^n term: $c^\dagger_{\cdot\uparrow}\, c^\dagger_{\cdot\downarrow}\, c_{\cdot\downarrow}\, c_{\cdot\uparrow}$

Useful starting point:

BCS Hamiltonian: $$H = \sum_{\underset{\sim}{k}\sigma} \varepsilon_{\underset{\sim}{k}}\, c^\dagger_{\underset{\sim}{k}\sigma} c_{\underset{\sim}{k}\sigma} - \frac{g}{L^d} \sum_{\underset{\sim}{k}\underset{\sim}{k}'\underset{\sim}{q}} c^\dagger_{\underset{\sim}{k}+\underset{\sim}{q}\uparrow}\, c^\dagger_{-\underset{\sim}{k}\downarrow}\, c_{-\underset{\sim}{k}'+\underset{\sim}{q}\downarrow}\, c_{\underset{\sim}{k}'\uparrow}$$

$\uparrow$ "small" $\to 0$

Mean field assumption:

assume that there is a state $|\Omega_s\rangle$ s.t.

$$\frac{g}{L^d}\sum_{\underset{\sim}{k}} \langle \Omega_s | c_{-\underset{\sim}{k}\downarrow} c_{\underset{\sim}{k}\uparrow} | \Omega_s \rangle = \Delta \quad \leftarrow \text{some } \#$$

$$\frac{g}{L^d}\sum_{\underset{\sim}{k}} \langle \Omega_s | c^\dagger_{\underset{\sim}{k}\uparrow} c^\dagger_{-\underset{\sim}{k}\downarrow} | \Omega_s \rangle = \bar{\Delta} \qquad (\#) \qquad \#^2 \sim 0$$

Approx$^{\text{n}}$ for mean field: $\sum_{\underset{\sim}{k}} c_{-\underset{\sim}{k}\downarrow} c_{\underset{\sim}{k}\uparrow} = \frac{L^d}{g}\Delta + \left(\sum_{\underset{\sim}{k}} c_{-\underset{\sim}{k}\downarrow} c_{\underset{\sim}{k}\uparrow} - \frac{L^d}{g}\Delta\right)$

BCS Ham in MF approx$^{\text{n}}$:

$$\hat{H} - \mu\hat{N} \simeq \sum_{\underset{\sim}{k}} \xi_{\underset{\sim}{k}} \sum_\sigma c^\dagger_{\underset{\sim}{k}\sigma} c_{\underset{\sim}{k}\sigma} - \left(\bar{\Delta} c_{-\underset{\sim}{k}\downarrow} c_{\underset{\sim}{k}\uparrow} + \Delta c^\dagger_{\underset{\sim}{k}\uparrow} c^\dagger_{-\underset{\sim}{k}\downarrow}\right) + \frac{L^d}{g}|\Delta|^2$$

$$\xi_{\underset{\sim}{k}} = \varepsilon_{\underset{\sim}{k}} - \mu$$

Make bilinear form explicit:

$$H-\mu N = \sum_{\underset{\sim}{k}} \left[\xi_{\underset{\sim}{k}} \left(c^\dagger_{\underset{\sim}{k}\uparrow} c_{\underset{\sim}{k}\uparrow} + c^\dagger_{+\underset{\sim}{k}\downarrow} c_{\underset{\sim}{k}\downarrow}\right) - \bar{\Delta}\, c_{-\underset{\sim}{k}\downarrow} c_{\underset{\sim}{k}\uparrow} - \Delta\, c^\dagger_{\underset{\sim}{k}\uparrow} c^\dagger_{-\underset{\sim}{k}\downarrow}\right] + \frac{L^d}{g}|\Delta|^2$$

$$\to \quad c^\dagger_{-\underset{\sim}{k}\downarrow} c_{-\underset{\sim}{k}\downarrow} = 1 - c_{-\underset{\sim}{k}\downarrow} c^\dagger_{-\underset{\sim}{k}\downarrow}$$

Use Nambu spinor $\Psi^\dagger_{\underset{\sim}{k}} = \left(c^\dagger_{\underset{\sim}{k}\uparrow} \quad c_{-\underset{\sim}{k}\downarrow}\right)$ $\quad \Psi_{\underset{\sim}{k}} = \begin{pmatrix} c_{\underset{\sim}{k}\uparrow} \\ c^\dagger_{-\underset{\sim}{k}\downarrow} \end{pmatrix}$

$$H-\mu N = \sum_{\underset{\sim}{k}} \left(c^\dagger_{\underset{\sim}{k}\uparrow} \quad c_{-\underset{\sim}{k}\downarrow}\right) \begin{pmatrix} \xi_{\underset{\sim}{k}} & -\Delta \\ -\bar{\Delta} & -\xi_{\underset{\sim}{k}} \end{pmatrix} \begin{pmatrix} c_{\underset{\sim}{k}\uparrow} \\ c^\dagger_{-\underset{\sim}{k}\downarrow} \end{pmatrix} + \sum_{\underset{\sim}{k}} \xi_{\underset{\sim}{k}} + \frac{L^d}{g}|\Delta|^2$$

Let's assume $\bar{\Delta} = \Delta \in \mathbb{R}$.

This can be diagonalized by a Bogoliubov tr^n:
(previous):

$$H = \begin{pmatrix} a & b \\ b & -a \end{pmatrix} \qquad U H U^\dagger = \begin{pmatrix} \varepsilon & 0 \\ 0 & -\varepsilon \end{pmatrix} \quad \text{with} \quad U = \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}$$

$$\& \quad \tan 2\theta = \frac{b}{a} \qquad \varepsilon = \left[a^2 + b^2\right]^{1/2}$$

check (exercise)

Explicitly here, we use a Bogoliubov $\forall\ \underset{\sim}{k}$

Write $\quad \chi_k \equiv U_k \Psi_k = \begin{pmatrix} \cos\theta_k & \sin\theta_k \\ \sin\theta_k & -\cos\theta_k \end{pmatrix} \begin{pmatrix} c_{\underset{\sim}{k}\uparrow} \\ c^{+}_{-\underset{\sim}{k}\downarrow} \end{pmatrix}$

$\to \begin{pmatrix} \alpha_{\underset{\sim}{k}\uparrow} \\ \alpha^{+}_{-\underset{\sim}{k}\downarrow} \end{pmatrix} \quad - \quad \alpha_{\underset{\sim}{k}\uparrow} = \cos\theta_{\underset{\sim}{k}}\, c_{\underset{\sim}{k}\uparrow} + \sin\theta_k\, c^{+}_{-\underset{\sim}{k}\downarrow}$

$\nwarrow$ obey CACR (exercise)

$\Psi_k^\dagger H_k \Psi_k = H_k U_k^\dagger \underbrace{U_k \Psi_k}$

Diagonal form at $\underset{\sim}{k}$ for $\tan 2\theta_k = -\Delta/\xi_k$

Energies: $\lambda_{\underset{\sim}{k}} \equiv \left[\Delta^2 + \xi_k^2\right]^{1/2}$

Hamiltonian:

$$H - \mu N = \sum_{\underset{\sim}{k}} \chi^\dagger_{\underset{\sim}{k}} \begin{pmatrix} \lambda_{\underset{\sim}{k}} & 0 \\ 0 & -\lambda_{\underset{\sim}{k}} \end{pmatrix} \chi_{\underset{\sim}{k}} + \sum_k \xi_k + \frac{L^d}{g}|\Delta|^2$$

$$= \sum_{\underset{\sim}{k}} \lambda_{\underset{\sim}{k}} \sum_\sigma \alpha^\dagger_{\underset{\sim}{k}\sigma} \alpha_{\underset{\sim}{k}\sigma} + \sum_{\underset{\sim}{k}} \left(\xi_{\underset{\sim}{k}} - \lambda_{\underset{\sim}{k}}\right) + \frac{L^d}{g}|\Delta|^2$$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \uparrow \qquad\qquad\qquad \uparrow$

State $|\Omega_s\rangle$: strange beast

Ground state of α's, we can write it as

$$|\Omega_s\rangle = \prod_{\underset{\sim}{k}} \alpha_{\underset{\sim}{k}\uparrow} \alpha_{-\underset{\sim}{k}\downarrow} |0\rangle = \prod_{\underset{\sim}{k}} \left(\cos\theta_{\underset{\sim}{k}} - \sin\theta_{\underset{\sim}{k}} c^{\dagger}_{\underset{\sim}{k}\uparrow} c^{\dagger}_{-\underset{\sim}{k}\downarrow} \right) |0\rangle$$

fermion vacuum: $c_{\underset{\sim}{k}\sigma}|0\rangle = 0$ esercizio

Check: is $|\Omega_s\rangle$ the vacuum of α's?

Well, $\alpha_{\underset{\sim}{k}\sigma}|\Omega_s\rangle = 0$

?

Last step: ensure self-consistency

$$\Delta \overset{?}{=} \frac{g}{L^d} \sum_{\underset{\sim}{k}} \langle \Omega_s | c_{-\underset{\sim}{k}\downarrow} c_{\underset{\sim}{k}\uparrow} | \Omega_s \rangle$$

But: $\begin{pmatrix} c_{k\uparrow} \\ c^\dagger_{-k\downarrow} \end{pmatrix} = U^{-1} \begin{pmatrix} \alpha_{k\uparrow} \\ \alpha_{-k\downarrow} \end{pmatrix}$

$\to c_{\underset{\sim}{k}\uparrow} = \cos\Theta_k \alpha_{\underset{\sim}{k}\uparrow} + \sin\Theta_{\underset{\sim}{k}} \alpha^\dagger_{-\underset{\sim}{k}\downarrow}$

$\to c_{-\underset{\sim}{k}\downarrow} = \sin\Theta_{\underset{\sim}{k}} \alpha^\dagger_{\underset{\sim}{k}\uparrow} - \cos\Theta_{\underset{\sim}{k}} \alpha_{-\underset{\sim}{k}\downarrow}$

$$= \frac{g}{L^d} \sum_{\underset{\sim}{k}} \langle \Omega_s | \underbrace{(\sin\Theta_k \alpha^\dagger_{\underset{\sim}{k}\uparrow} - \cos\Theta_k \alpha_{-\underset{\sim}{k}\downarrow})}_{0} \underbrace{(\cos\Theta_k \alpha_{k\uparrow} + \sin\Theta_k \alpha^\dagger_{-\underset{\sim}{k}\downarrow})}_{\to 0} | \Omega_s \rangle$$

$$= -\frac{g}{L^d} \sum_{\underset{\sim}{k}} + \sin\Theta_k \cos\Theta_k \underbrace{\langle \Omega_s | \alpha_{-k\downarrow} \alpha^\dagger_{-\underset{\sim}{k}\downarrow} | \Omega_s \rangle}_{1-\alpha^\dagger\alpha\ = 1}$$

But: from Bogoliubov,

$$\sin 2\Theta_k = -\frac{\Delta}{\lambda_{\underset{\sim}{k}}} \quad \left(\cos 2\Theta_k = \frac{\xi_k}{\lambda_k}\right) \quad \text{(check: exercise)}$$

$$\sin 2\Theta = 2\sin\Theta\cos\Theta$$

so $$\Delta \overset{?}{=} +\frac{g}{2L^d} \sum_{\underset{\sim}{k}} \frac{\Delta}{[\Delta^2+\xi^2]^{1/2}} =$$

$$\Delta = \frac{g}{2L^d} \sum_{k} \frac{\Delta}{\left[\Delta^2 + \xi_k^2\right]^{1/2}} = \frac{g\Delta}{2} \int_{-\omega_D}^{\omega_D} d\xi \, \frac{\nu(\xi)}{\left[\Delta^2 + \xi^2\right]^{1/2}}$$

Approx^{m}: density of states $\nu(\xi) \sim$ cst ν

so $$\Delta = g\nu\Delta \int_0^{\omega_D} d\xi \, \frac{1}{\left[\Delta^2+\xi^2\right]^{1/2}} = g\Delta\nu \int_0^{\operatorname{arcsh}\frac{\omega_D}{\Delta}} dx = g\Delta\nu \operatorname{asinh}\frac{\omega_D}{\Delta}$$

To perform integral: $\xi = \Delta \operatorname{sh} x$ $\quad \left[\Delta^2+\xi^2\right]^{1/2} = \Delta \cosh x$

$$d\xi = \Delta \operatorname{ch} x \, dx$$

Self-consistency: $$\Delta = \frac{\omega_D}{\operatorname{sh}(1/g\nu)} \underset{g\nu \ll 1}{\simeq} 2\omega_D e^{-1/g\nu}$$

so self-consistent theory if $\Delta = 2\omega_D e^{-1/g\nu}$

"assumed" $\quad$ calculated

Density of states of physical exc$^{\underset{\frown}{n}}$s

$$\rho(\varepsilon) = \frac{1}{L^d} \sum_{\underset{\sim}{k}\sigma} \delta(\varepsilon - \lambda_{\underset{\sim}{k}}) = \int d\xi \; \underbrace{\frac{1}{L^d} \sum_{\underset{\sim}{k}\sigma} \delta(\xi - \xi_{k})}_{} \; \delta(\varepsilon - \lambda(\xi))$$

DOS of original fermions $\nu(\xi) \sim \nu$ (constant)

so $\rho(\varepsilon) = \nu \int d\xi \, \delta(\varepsilon - \lambda(\xi))$ $\qquad \lambda(\xi) = [\Delta^2 + \xi^2]^{1/2}$

Use $\delta(\zeta(\xi)) = \sum_{\text{zeros of } \zeta \; i} \frac{\delta(\xi - \xi_i)}{|\zeta'(\xi_i)|}$

Here, $\rho(\varepsilon) = \nu \sum_{s=\pm 1} \frac{\delta(\xi - s[\varepsilon^2 - \Delta^2]^{1/2})}{|\xi| / [\Delta^2 + \xi^2]^{1/2}} = \frac{2\nu\varepsilon}{[\varepsilon^2 - \Delta^2]^{1/2}} \Theta(\varepsilon - \Delta)$

heaviside f^{n}

Plot: