$$\Theta(t) = \begin{cases} 1 & t>0 \\ (1/2 & t=0) \\ 0 & t<0 \end{cases}$$

Thermal correl$^{\underline{n}}$ f$^{\underline{n}}$s

$$\mathcal{C}_{ret}^{\hat{O},\hat{P}}(t) = \frac{1}{Z}\sum_\alpha \mathcal{C}_{ret,\psi_\alpha}^{\hat{O},\hat{P}}\, e^{-\beta E_\alpha} = \frac{-i\,\Theta(t)}{Z}\sum_\alpha \langle\psi_\alpha|\left[\hat{O}^I(t),\hat{P}^I(0)\right]|\psi_\alpha\rangle e^{-\beta E_\alpha}$$

$$Z = \sum_\alpha e^{-\beta E_\alpha}$$

Do just as for the retarded f$^{\underline{n}}$: introduce $\mathbb{1}$, do FT to ω

$$\Rightarrow \quad \mathcal{C}_{ret}^{\hat{O},\hat{P}}(\omega) = \frac{1}{Z}\sum_{\alpha,\alpha'} O_{\alpha\alpha'} P_{\alpha'\alpha}\, \frac{e^{-\beta E_\alpha} - e^{-\beta E_{\alpha'}}}{\omega + E_\alpha - E_{\alpha'} + i\eta}$$

Similar expressions for advanced & real-time f$^{\underline{n}}$s,

see notes eq$^{\underline{n}}$s 7.55 & 7.56

Yet another correl^{n}: the imaginary time one

$$\mathcal{C}_{\tau}^{\hat{O},\hat{P}}(\tau_1-\tau_2) \equiv -\left\langle T_{\tau}\left\{\hat{O}(\tau_1)\,\hat{P}(\tau_2)\right\}\right\rangle$$

$$O(\tau) \equiv e^{\tau H_0}\, O\, e^{-\tau H_0}$$

Lehmann repres^{n}:

$$\mathcal{C}_{\tau}^{\hat{O},\hat{P}}(\tau) = -\frac{1}{Z}\sum_{\alpha,\alpha'} O_{\alpha\alpha'} P_{\alpha'\alpha}\, e^{(E_\alpha - E_{\alpha'})\tau}\left\{\Theta(\tau)e^{-\beta E_\alpha} + \Theta(-\tau)e^{-\beta E_{\alpha'}}\right\}$$

(for $\tau \in [-\beta, \beta]$)

Nice property: $\mathcal{C}_{\tau}^{\hat{O},\hat{P}}(\tau) = \mathcal{C}_{\tau}^{\hat{O},\hat{P}}(\tau+\beta)$

$\longrightarrow$ This f^{n} can be decomposed in Matsubara freq.

FT to Matsubara using $\mathcal{C}(i\omega_n) = \int_0^\beta d\tau\, e^{i\omega_n \tau}\, \mathcal{C}(\tau)$

Get the Lehmann repres^{n}

$$\mathcal{C}_{\tau}^{\hat{O},\hat{P}}(i\omega_n) = \frac{1}{Z}\sum_{\alpha\alpha'} O_{\alpha\alpha'} P_{\alpha'\alpha} \frac{e^{-\beta E_\alpha} - e^{-\beta E_{\alpha'}}}{i\omega_n + E_\alpha - E_{\alpha'}}$$

Useful identity:

$$\mathcal{C}_{ret}^{\hat{O},\hat{P}}(\omega) = \mathcal{C}_{\tau}^{\hat{O},\hat{P}}(i\omega_n)\Big|_{i\omega_n = \omega + i\eta}$$

"Master" f^{n}:

$$\mathcal{C}_{\tau}^{\hat{O},\hat{P}}(z) = \frac{1}{Z}\sum_{\alpha\alpha'} \frac{e^{-\beta E_\alpha} - e^{-\beta E_{\alpha'}}}{z + E_\alpha - E_{\alpha'}} O_{\alpha\alpha'} P_{\alpha'\alpha}$$

"one f^{n} to rule them all"

$\mathcal{C}_{ret}$: $z \to \omega + i\eta$

$\mathcal{C}_{adv}$: $z \to \omega - i\eta$

$\mathcal{C}_{\tau}$: $z \to i\omega_n$

Concrete example: correl$^{\underline{n}}$ f$^{\underline{n}}$ for free particles

Free fermions $\quad H_0 = \sum_k \varepsilon_k a_k^\dagger a_k \qquad \{a_k, a_{k'}^\dagger\} = \delta_{kk'}$

Retarded f$^{\underline{n}}$ $\quad \mathcal{C}^{ret}_{\beta,\mu;k}(t_1-t_2) = -i\Theta(t_1-t_2)\langle\{a_k(t_1), a_k^\dagger(t_2)\}\rangle_{\beta,\mu}$

$$\langle\ldots\rangle_{\beta,\mu} = \frac{1}{Z_{\beta,\mu}}\sum_\alpha (\ldots)\, e^{-\beta(E_\alpha-\mu N_\alpha)} \qquad Z_{\beta,\mu} \equiv \sum_\alpha e^{-\beta(E_\alpha-\mu N_\alpha)}$$

Treat time dep. of operators thanks to Heisenberg:

$$a_k(t) = e^{i(H_0-\mu N)t} a_k e^{-i(H_0-\mu N)t} = e^{-i(\varepsilon_k-\mu)t} a_k = e^{-i\xi_k t}$$

$\xi_k \equiv \varepsilon_k - \mu$

$$e^{\alpha\hat b^\dagger \hat b}\, \hat b\, e^{-\alpha\hat b^\dagger \hat b} = e^{-\alpha}\hat b \qquad a_k^\dagger(t) = e^{i\xi_k t} a_k^\dagger$$

so $\quad \mathcal{C}^{ret}_{\beta,\mu;k}(t_1-t_2) = -i\Theta(t_1-t_2)\, e^{-i\xi_k(t_1-t_2)}\left\{\underbrace{\langle a_k a_k^\dagger\rangle_{\beta,\mu} + \langle a_k^\dagger a_k\rangle_{\beta,\mu}}_{1}\right\}$

$$= -i\Theta(t_1-t_2)\, e^{-i\xi_k(t_1-t_2)}$$

Fourier tr^{m}: $\mathcal{C}(\omega) \equiv \int_{-\infty}^{\infty} dt\, e^{i\omega t - \eta|t|}\, \mathcal{C}(t)$

so $\mathcal{C}^{ret}_{\beta,\mu;k}(\omega) = -i \underbrace{\int_{-\infty}^{\infty} dt\, \Theta(t)}_{\int_0^\infty dt} e^{i\omega t - \eta|t| - i\xi_k t} = \frac{1}{\omega - \xi_k + i\eta}$

$$\boxed{\mathcal{C}^{ret}_{\beta,\mu;k}(\omega) = \frac{1}{\omega - \xi_k + i\eta}}$$

Advanced f^{n}: $\mathcal{C}^{adv} = i\Theta(t_2 - t_1)\ldots \longrightarrow \frac{1}{\omega - \xi_k - i\eta}$

Other useful f^{ns}: "greater": $\mathcal{C}^{>}(t) \equiv -i\langle a_k(t) a^{+}_k(0)\rangle$

$\mathcal{C}^{<}(t) \equiv -i\underset{\substack{+1\ B\\ -1\ F}}{\varsigma}\langle a^{+}_k(0)\, a_k(t)\rangle$

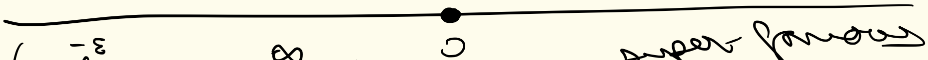
Rel^{ns}: $\mathcal{C}^{ret}(t) = \Theta(t)\left(\mathcal{C}^{>} - \mathcal{C}^{<}\right)$

$\mathcal{C}^{adv}(t) = \Theta(-t)\left(\mathcal{C}^{<} - \mathcal{C}^{>}\right)$ (see notes)

One last thing: in calculations using Kubo one often ends up with the combination of $\mathcal{G}^{ret} - (\mathcal{G}^{ret})^*$

→ useful qtty $A_{\beta,\mu;k}(\omega) \equiv -2\,\mathrm{Im}\left\{\mathcal{G}^{ret}_{\beta,\mu;k}(\omega)\right\}$

Dirac identity: $\lim\limits_{\eta\to 0^+} \dfrac{1}{\omega \pm i\eta} = \mp i\pi\,\delta(\omega) + P\,\dfrac{1}{\omega}$



$$P\int_{-\infty}^{\infty} dx\, f(x) = \lim_{\varepsilon\to 0}\left(\int_{-\infty}^{-\varepsilon} dx + \int_{\varepsilon}^{\infty} dx\right) f(x)$$

super famous SPECTRAL F^n

Use this with $\mathcal{G}^{ret} = \dfrac{1}{\omega-\xi+i\eta} \longrightarrow A_{\beta,\mu;k}(\omega) = 2\pi\,\delta(\omega-\xi_k)$

Photoemission spectroscopy (from Final 2016)

$$H_0 = H_c - \mu N_c + H_s + H_a$$

(lattice) $H_c - \mu N_c = \sum_k \xi_k c_k^\dagger c_k$ $\qquad \{c_k, c_{k'}^\dagger\} = \delta_{kk'}$

(continuum) $H_s = \int_{-\infty}^{\infty} dp \, \varepsilon_s(p) \, s^\dagger(p) s(p)$ $\qquad \{s(p), s^\dagger(p')\} = \delta(p-p')$

(monochromatic light field) $H_a = \omega a^\dagger a$ $\qquad [a, a^\dagger] = 1$

Coupling: $H_{sca} = \gamma \int_{-\infty}^{\infty} dp \sum_k s^\dagger(p) c_k a + h.c.$

Full Hamiltonian: $H = H_0 + H_{sca}$

Observable: rate (number per unit time) of scatterings $c \to s$

$$\hat{O} = \frac{d}{dt} n_s(p)$$

By Heisenberg eq^n of motion:

$$\frac{d}{dt} n_s(p) = i\left[H, \underbrace{n_s(p)}_{s^\dagger(p)s(p)}\right]$$

Calculate: $$i\left[H, n_s(p)\right] = \underbrace{i\left[H_0, n_s(p)\right]}_{0} + i\left[H_{sca}, n_s(p)\right]$$

since H_0 conserves # of c, s, a independently.

$$i\left[H_{sca}, n_s(p)\right] = i\gamma \int_{-\infty}^{\infty} dp' \sum_k \underbrace{\left[s^\dagger(p')c_k a, s^\dagger(p)s(p)\right]}\ + \text{h.c.}$$

But $$\left[s^\dagger(p')c_k a, s^\dagger(p)s(p)\right] = s^\dagger(p')\underbrace{c_k a\, s^\dagger(p)s(p)}_{s^\dagger(p)s(p)c_k a} - s^\dagger(p)s(p)s^\dagger(p')c_k a$$

$$= \left[\underbrace{s^\dagger(p')s^\dagger(p)}_{-s^\dagger(p)s^\dagger(p')}s(p) - s^\dagger(p)s(p)s^\dagger(p')\right]c_k a = -s^\dagger(p)\underbrace{\left[s^\dagger(p')s(p) + s(p)s^\dagger(p')\right]}_{\delta(p-p')}c_k a$$

$\{s(p), s^\dagger(p')\}$

$$= -\delta(p-p')\, s^\dagger(p)c_k a$$

so $i\left[H_{sca}, n_s(p)\right] = -i\gamma \int_{-\infty}^{\infty} dp' \sum_k \delta(p-p') s^\dagger(p) c_k a \quad + h.c.$

$$= \underbrace{-i\gamma \sum_k s^\dagger(p) c_k a}_{} + i\gamma^* \sum_k a^\dagger c_k^\dagger s(p)$$

b) Assume light field comes from a coherent laser, so in coherent state $|\phi\rangle_a = e^{\phi a^\dagger}|0\rangle_a$

Property: $a|\phi\rangle_a = \phi|\phi\rangle_a$ (see useful formulas)

For e^-s, get effective theory by taking expectation value over this light field

$$H_{sc}(t) = \frac{{}_a\langle\phi(t)| H_{sca} |\phi(t)\rangle_a}{{}_a\langle\phi|\phi\rangle_a} \leftarrow \quad |\phi(t)\rangle_a = e^{-iH_a t}|\phi\rangle_a$$

→ calculate this. Only thing needed: $\dfrac{{}_a\langle\phi(t)| a |\phi(t)\rangle_a}{{}_a\langle\phi|\phi\rangle_a}$ $\quad H_{sca} = (\ldots) a + h.c.$

Need $$ {}_a\langle\phi(t)|H_{sca}|\phi(t)\rangle_a = \int dp \sum_k s^\dagger(p) c_k \; \underbrace{{}_a\langle\phi(t)|a|\phi(t)\rangle_a}_{\circledast} $$

$$ {}_a\langle\phi(t)|a|\phi(t)\rangle_a = {}_a\langle\phi|\underbrace{e^{iH_at}\, a\, e^{-iH_at}}|\phi\rangle_a $$

$$ = e^{-i\omega t} a \quad \text{since } H_a = \omega a^\dagger a $$

$$ = e^{-i\omega t}\, {}_a\langle\phi|\underbrace{a|\phi\rangle_a}_{\phi|\phi\rangle_a} = \phi e^{-i\omega t}\, {}_a\langle\phi|\phi\rangle_a $$

so $$ H_{sc}(t) = \gamma\phi e^{-i\omega t} \int_{-\infty}^{\infty} dp \sum_k s^\dagger(p) c_k \equiv \gamma\phi e^{-i\omega t} J + \text{h.c.} $$

$+\text{h.c.}$

$$ J \equiv \int_{-\infty}^{\infty} dp\, j(p) \qquad j(p) \equiv \sum_k s^\dagger(p) c_k $$

observable

Also, $$ R(p,t) \equiv \frac{{}_a\langle\phi(t)|\frac{d}{dt} n_s(p)|\phi(t)\rangle_a}{{}_a\langle\phi|\phi\rangle_a} = -i\gamma\phi e^{-i\omega t} j(p) + \text{h.c.} $$

c) Now: $H(t) = H_{0,sc} + H_{sc}(t)$ $\qquad H_{0,sc} = H_c - \mu N_c + H_s$

Calculate the response in Kubo linear resp. formalism

on ~~state~~ $|\mu;0\rangle \equiv |\mu\rangle_c \otimes |0\rangle_s$ vacuum of S

c fermions at 0 temp. & chem. p. μ

$$\langle(\ldots)\rangle \equiv \langle\mu;0|(\ldots)|\mu,0\rangle$$

Q: apply the Kubo formula to get $\bar{R}(p,t)$

Observable: $\hat{O} = -i\gamma\phi e^{-i\omega t} j(p) + h.c.$ $\quad s^+c \quad c^+s$

Perturbation: $F(t)\hat{P} = \underbrace{\gamma\phi e^{-i\omega t}}_{F(t)} \underbrace{J}_{\hat{P}} + \underbrace{\gamma^*\phi^* e^{i\omega t}}_{F^*(t)} \underbrace{J^+}_{P^+}$

$\langle s^+c\, s^+c\rangle \to 0$

$\langle cc\rangle = 0$

$\langle cc^+\rangle \neq 0$

$\langle c^+c\rangle \neq 0$

Kubo:

$$\bar{R}(p,t) = \overset{\langle R\rangle}{\bar{R}_0} - i|\gamma|^2|\phi|^2 \int_{-\infty}^{\infty} dt'\, \underbrace{C_{ret}^{j(p),J^+}(t-t')}\, e^{-i\omega(t-t')} + h.c.$$

$$\equiv -i\Theta(t-t')\left\langle\left[j^I(p,t), J^{I^+}(t')\right]\right\rangle$$

d) Retarded f^{m}_{r}.

$$C^{j,j^\dagger}_{ret} = -i\,\Theta(t-t')\int_{-\infty}^{\infty} dp' \langle [j(p,t), j^\dagger(p',t')] \rangle$$

$$= -i\,\Theta(t-t')\int_{-\infty}^{\infty} dp' \sum_{k,k'} \underbrace{\langle [S^\dagger(p,t)c_k(t), c^\dagger_{k'}(t')S(p',t')] \rangle}_{①-②}$$

$$① = \langle S^\dagger(p,t)c_k(t)c^\dagger_{k'}(t')S(p',t')\rangle \underset{\text{Wick's}}{=} \underbrace{\langle S^\dagger(p,t)S(p',t')\rangle}\langle c_k(t)c^\dagger_{k'}(t')\rangle$$

$$\langle \underbrace{S^\dagger(p,t)}_{e^{i\varepsilon_s t}S^\dagger(p)}\underbrace{S(p',t')}_{e^{-i\varepsilon_s t'}S(p')}\rangle = e^{i\varepsilon_s(t-t')}\underbrace{\langle S^\dagger(p)S(p')\rangle}_{0} = 0$$

vacuum of S.

$$\langle \underbrace{c_k(t)}_{e^{-i\xi_k t}c_k}\underbrace{c^\dagger_{k'}(t')}_{e^{i\xi_{k'}t'}c^\dagger_{k'}}\rangle = e^{-i\xi_k t+i\xi_{k'}t'}\underbrace{\langle c_k c^\dagger_{k'}\rangle}_{\delta_{kk'}(1-\bar{n}_k)} = e^{-i\xi_k(t-t')}(1-\bar{n}_k)$$

← see q^{m}_{r}.

$$② = \langle c^\dagger_{k'}(t') S(p',t') S^\dagger(p,t) c_k(t) \rangle$$

$$= \underbrace{\langle c^\dagger_{k'}(t') c_k(t) \rangle}_{ⓐ} \underbrace{\langle S(p',t') S^\dagger(p,t) \rangle}_{ⓑ}$$

$$c^\dagger_k(t) = e^{i(H_c - \mu N_c)t} c^\dagger_k e^{-i()}$$

$$= e^{i\xi_k t} c^\dagger_k$$

$$ⓐ = \underbrace{e^{i\xi_{k'} t' - i\xi_k t}} \underbrace{\langle c^\dagger_{k'} c_k \rangle}$$

$$\cdot \delta_{kk'} \bar{n}_k$$

$$ⓑ \ \langle S(p',t') S^\dagger(p,t) \rangle = e^{-i\varepsilon_s(p)t' + i\varepsilon_s(p)t} \langle \underbrace{S(p') S^\dagger(p)}_{\delta(p-p') - S^\dagger(p)S(p')} \rangle =$$

$$\doteq \delta(p-p') e^{+i\varepsilon_s(p)(t-t')}$$

$$\text{So } \mathcal{C}^{j,J^\dagger}_{ret} = +i\Theta(t-t') \sum_k e^{i(\varepsilon_s(p) - \xi_k)(t-t')} \bar{n}_k$$

answer to d).

e) $\overline{R}(p,t) = -i|\gamma|^2|\phi|^2 \int_{-\infty}^{\infty} dt' \, \underline{C^{j(p),J^+}_{ret}}(t-t') \underline{e^{-i\omega(t-t')}} + h.c.$

$\equiv -i\,\underbrace{\theta(t-t')}\langle[j,J^+]\rangle$

$$= |\gamma|^2|\phi^2| \sum_k \overline{n}_k \underbrace{\int_{-\infty}^{t} dt' \, e^{i(\xi_s^{(p)}-\xi_k-\omega)(t-t')}}$$

$$\int_{-\infty}^{0} dt' \, e^{-i[\xi_s(p)-\xi_k-\omega]t'}$$

Time integ:

$$\int_{-\infty}^{0} dt' e^{-i\Omega t' + \eta t'} = \frac{e^{\cdots}}{-i[\Omega+i\eta]}\Bigg|_{-\infty}^{0} = \frac{i}{\Omega+i\eta}$$

so $$\overline{R}(p,t) = |\gamma|^2|\phi|^2 \sum_k \overline{n}_k \underbrace{\left\{ \frac{i}{\omega-\xi_s(p)+\xi_k+i\eta} - \frac{i}{\omega-\xi+\xi-i\eta} \right\}}$$

indep of t!

$$\frac{2\eta}{(\omega-\xi+\xi)^2+\eta^2} \to 2\pi\delta(\omega-)$$

$\overline{R}(p,t)$

$\downarrow$ t-indep.

so $$\overline{R}(p) = 2\pi|\gamma|^2|\phi|^2 \sum_k \delta(\omega-\xi_s(p)+\xi_k)\overline{n}_k$$ $\leftarrow$ spectral fn.