Lecture 5a

$$\hat{H}|n\rangle = E_n|n\rangle$$

$$e^{i/\hbar \hat{H} t}|\psi(0)\rangle = |\psi(t)\rangle$$

Propagator: $\quad \mathcal{G}(x', t'; x, t) = \langle x'| e^{-\frac{i}{\hbar}\hat{H}(t'-t)} |x\rangle$

$$|\mathcal{G}|^2 = |\langle x'|\hat{u}|x\rangle|^2 = |\langle x'|\psi(t'-t)\rangle|^2 \qquad = \langle vac|\hat{c}_{x'}(t')\hat{c}^{\dagger}_{x}(t)|vac\rangle$$

$$|\psi(t)\rangle \equiv |x\rangle$$

$$\psi(x',t') = \langle x' | \psi(t') \rangle$$

$$= \langle x' | e^{-i/\hbar \hat{H}(t'-t)} | \psi(t) \rangle$$

$$= \int dx \, \langle x' | e^{-\frac{i}{\hbar}\hat{H}(t'-t)} | x \rangle \langle x | \psi(t) \rangle$$

$$\boxed{\psi(x',t') = \int dx \, G(x',t';x,t)\, \psi(x,t)}$$

$$\int dx \, |x\rangle\langle x| = \hat{I}$$

$$\langle x' | e^{-\frac{i}{\hbar}\hat{H}(t'-t)} | x \rangle = \sum_n \langle x' | e^{-\frac{i}{\hbar}\hat{H}(t'-t)} | n \rangle \langle n | x \rangle$$

$$= \sum_n \langle x' | e^{-\frac{i}{\hbar}E_n(t'-t)} | n \rangle \langle n | x \rangle$$

$$= \sum_n e^{-i/\hbar\, E_n (t'-t)} \psi_n(x') \psi_n^*(x)$$

$|n\rangle$

$\langle x | n \rangle \equiv \psi_n(x)$

$$f(t) = \int d\omega\, e^{-i\omega t} \tilde{f}(\omega)$$

$$\hat{H} = \frac{\hat{p}^2}{2m} + V(\hat{x})$$

$$e^{-\frac{i}{\hbar}\left(\frac{\hat{p}^2}{2m} + V(\hat{x})\right)\Delta t} = e^{-\frac{i}{\hbar}\frac{\hat{p}^2}{2m}\Delta t} \, e^{-\frac{i}{\hbar}V(\hat{x})\Delta t}$$

$$+ O\left([\hat{x},\hat{p}]\,\Delta t^2\right)$$

Campbell-Hausdorf:

$$e^{\hat{A}+\hat{B}} = e^{\hat{A}} e^{\hat{B}} + \frac{1}{2}[A,B] + \ldots\ldots$$

$$\lim_{\Delta t \to 0}$$

$$\mathcal{G}(x't';xt) = \int dp \;\langle x'|p\rangle\langle p|\, e^{-\frac{i}{\hbar}\frac{\hat{p}^2}{2m}\Delta t}\, e^{-\frac{i}{\hbar}V(\hat{x})\Delta t}\,|x\rangle$$

$t'-t \equiv \Delta t$

$$= \int dp \;\langle x'|p\rangle\langle p|x\rangle\, e^{-i/\hbar \underbrace{\left(\frac{p^2}{2m}+V(x)\right)}_{H[p,x]}\Delta t}$$

classical

$$G(x't', xt) = \langle x' | \left(e^{-\frac{i}{\hbar}\hat{H}\Delta t} \right)^N | x \rangle$$

$t' - t = N\Delta t$

with

$\Delta t \to 0$

$N \to \infty$

$$= \langle x' | \hat{I} \; e^{-\frac{i}{\hbar}\frac{\hat{p}^2}{2m}\Delta t} \; e^{-\frac{i}{\hbar}V(\hat{x})\Delta t} \; \hat{I} \; e^{-\frac{i}{\hbar}\frac{\hat{p}^2}{2m}\Delta t} \cdots\cdots | x \rangle$$

$$\hat{I} = \int dx_n \int dp_n \; | x_n \rangle \underbrace{\langle x_n | p_n \rangle}_{\frac{1}{2\pi\hbar} e^{\frac{i}{\hbar}p_n x_n}} \langle p_n |$$

$$G = \int \left(\prod_{n=1}^{N-1} dx_n \right) \left(\prod_{n=1}^{N} dp_n \frac{1}{2\pi\hbar} \right) e^{-\frac{i}{\hbar}\Delta t \sum_{n=0}^{N-1} \left[\frac{p_{n+1}^2}{2m} + V(x_n) - p_{n+1} \frac{x_{n+1} - x_n}{\Delta t} \right]}$$

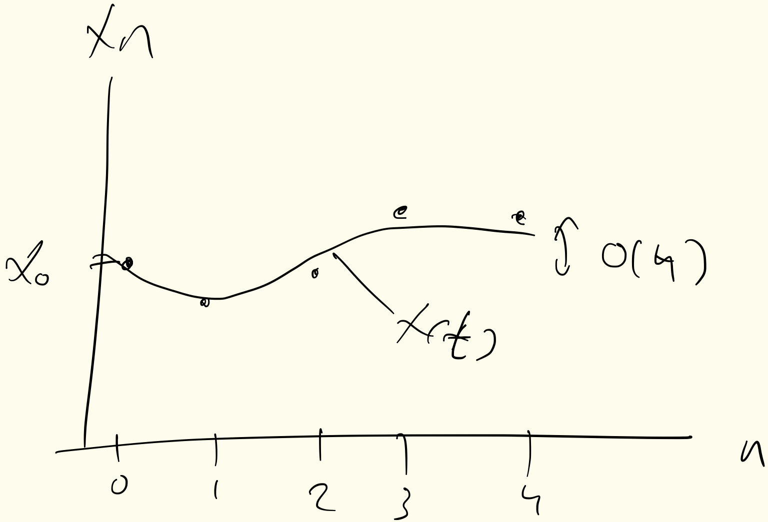
$$\Delta t \to 0$$

$$\sum_{n=0}^{N-1} \Delta t \longrightarrow \int dt$$

$$\frac{x_{n+1} - x_n}{\Delta t} \longrightarrow \frac{\partial}{\partial t} x(t) = \dot{x}$$

$$\lim_{N\to\infty} \int \left(\prod dx_n\right)\left(\prod dp_n \frac{1}{2\pi\hbar}\right) \equiv \int Dx\, Dp$$

$$G = \int \mathcal{D}p\, \mathcal{D}x \; e^{\frac{i}{\hbar}\int_t^{t'} dt'' \left(p\dot{x} - H[p,x]\right)}$$

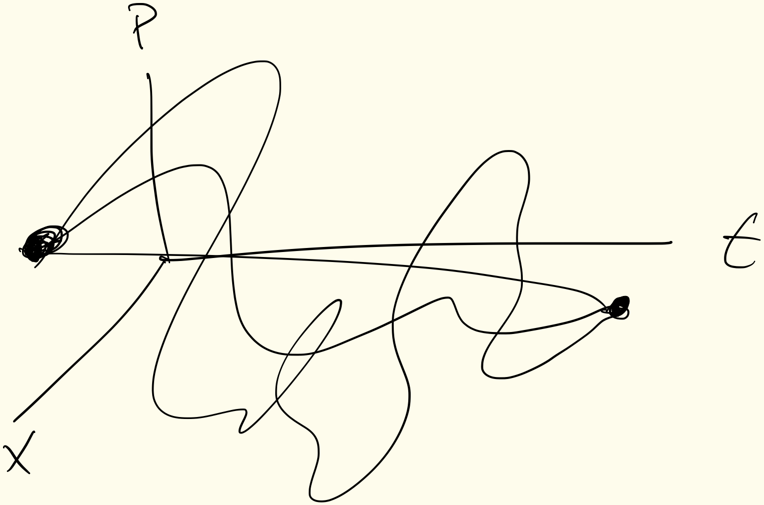


$$x_n \to x(t)\big|_{t_n}$$

$$e^{\frac{i}{\hbar}\Delta t \; p_{n+1} \frac{x_{n+1}-x_n}{\Delta t}}$$

$$p_{n+1}\left(x_{n+1}-x_n\right) \sim O(\hbar)$$

$$G = \int Dp\, Dx\, \underbrace{e^{-\frac{i}{\hbar}\int_t^{t'} dt'' \left[p\dot{x} - H \right]}}_{\text{amplitude}}$$



Dirac

"Sum over histories"

Lecture 5b

$$\mathcal{G} = \int \mathcal{D}x\, \mathcal{D}p \; e^{\frac{i}{\hbar}\int dt\,[p\dot{x} - H]}$$

$$H = \frac{p^2}{2m} + V(x)$$

$$= \int \mathcal{D}x \; e^{-\frac{i}{\hbar}\int dt\, V[x]} \underbrace{\int \mathcal{D}p \; e^{-\frac{i}{\hbar}\int dt\left(\frac{p^2}{2m} - p\dot{x}\right)}}$$

$$\int_{-\infty}^{\infty} dx\, e^{-\frac{a^2}{2}x^2} = \sqrt{\frac{2\pi}{a}}$$

$y = x + b/a$

$$\int_{-\infty}^{\infty} dx\, e^{-\frac{a^2}{2}x^2 + bx} = \int_{-\infty}^{\infty} dy\, e^{-\frac{a^2}{2}y^2 + \frac{b^2}{2a}}$$

$$= \sqrt{\frac{2\pi}{a}}\, e^{b^2/2a}$$

$$\int dz \; e^{-z^* \omega z} = \pi/\omega \qquad \int dw\, e^{-w \hat{D} w}$$

$$\int dx \int dy$$

$$z = x + iy$$

$$\hat{A} = \hat{O}^{T} \hat{D} \hat{O}$$

$$\vec{w} = \hat{O} \vec{v}$$

$$\int d\vec{v} \; e^{-\frac{1}{2} \vec{v}^{T} \hat{A} \vec{v}} = \underline{\underline{(2\pi)}}^{N/2} \left(\det(A) \right)^{-1/2}$$

V real, N components $\qquad$ A $N \times N$, real, symmetric

$$\int d\vec{v} \; e^{-\frac{1}{2}\vec{v}^T \hat{A} \vec{v} + \vec{J}^T \vec{v}} = (2\pi)^{N/2} \left(\det(A)\right)^{-1/2} e^{\frac{1}{2}\vec{J}^T A^{-1} \vec{J}}$$

$$\int d\vec{z} \; e^{-\vec{z}^\dagger \hat{A} \vec{z} + \vec{w}^\dagger \vec{z} + \vec{z}^\dagger \vec{u}} = \pi^N \left(\det(A)\right)^{-1} e^{\vec{w}^\dagger \hat{A}^{-1} \vec{u}}$$

$$\langle Z_m^* Z_n \rangle = A^{-1}_{nm}$$

$$\boxed{\langle A \rangle = \frac{1}{Z} \int d\psi \; e^{-\beta\psi} \hat{A}}$$

$$\langle \ldots \rangle \equiv \pi^{-N} \underline{\det(A)} \int dz \; e^{-z^\dagger A z} (\ldots)$$

$$\langle z^*_{i_1} z^*_{i_2} z^*_{i_3} \cdots z^*_{i_n} \, z_{j_1} z_{j_2} \cdots z_{j_n} \rangle$$

$$= A^{-1}_{j_1 i_1} A^{-1}_{j_2 i_2} A^{-1}_{j_3 i_3} \cdots$$

$$+ A^{-1}_{j_1 i_2} A^{-1}_{j_2 i_1} A^{-1}_{j_3 i_3} \cdots$$

$$+ \cdots$$

$$= \sum_{\text{PAIRINGS}(i,j)}$$

Wick's theorem

$$G = \int \mathcal{D}x \; e^{-\frac{i}{\hbar}\int dt \; V[x]} \int \mathcal{D}p \; e^{-\frac{i}{\hbar}\int dt \left(\frac{p^2}{2m} - p\dot{x}\right)}$$

$$= \lim_{N\to\infty} \int \left(\prod_{n=1}^{N-1} dx_n\right) \left(\frac{Nm}{i 2\pi \hbar (t-t')}\right)^{N/2} e^{\frac{i}{\hbar}\int_{t}^{t'} dt''}$$

$$\left(\frac{m\dot{x}^2}{2} - V[x(t'')]\right)$$

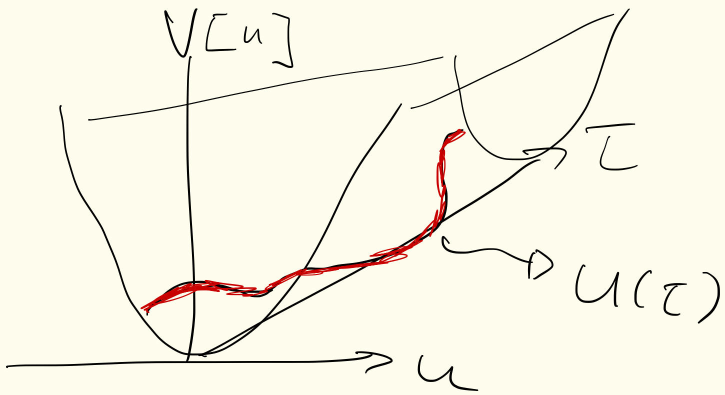
$$\sqrt{\frac{2\pi}{a}} \; e^{-b^2/a}$$

$$\Rightarrow \boxed{G = \int \mathcal{D}x \; e^{\frac{i}{\hbar} S[x, \dot{x}]}}$$

Lecture 5c

$$\mathcal{G} = \int \mathcal{D}x \; e^{\frac{i}{\hbar} S[x,\dot{x}]}$$

$$S[x,\dot{x}] = \int dt \left(\frac{m}{2}\left(\frac{\partial}{\partial t}x\right)^2 - V[x] \right)$$



$$H[u] = \int_0^{\cancel{t}\,L} d\tau \left(V[u] + \frac{\sigma}{2}\left(\frac{\partial}{\partial\tau}u\right)^2 \right)$$

$$\mathcal{G}_{\text{quantum}} = \int Dx \; e^{\frac{i}{\hbar}\int_0^t dt' \left(\frac{m}{2} (\partial_{t'} x)^2 - V[x] \right)}$$

$X(t)$ 0+1 D

$\downarrow$

$u(\tau, t)$ 1+1 D

$$Z_{\text{classical}} = \int Du \; e^{-\beta \int_0^L d\tau \left(\frac{\sigma}{2} (\partial_\tau u)^2 + V[u] \right)}$$

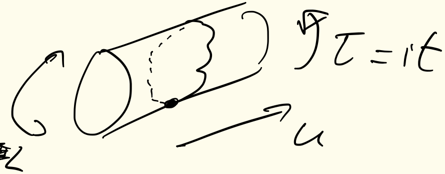
$\boxed{\text{1D classical} = \text{0D QM} \quad L \propto \beta}$

$$X \longleftrightarrow u$$

$$\underline{\underline{t'}} \longleftrightarrow \underline{\underline{-i\tau}} \quad \frac{1}{\beta\hbar} \frac{m}{\sigma}$$

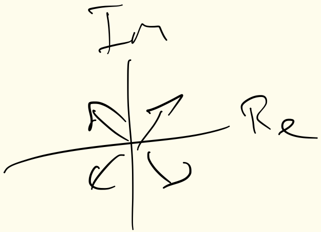
Periodic BC:

$u(L) = u(0)$



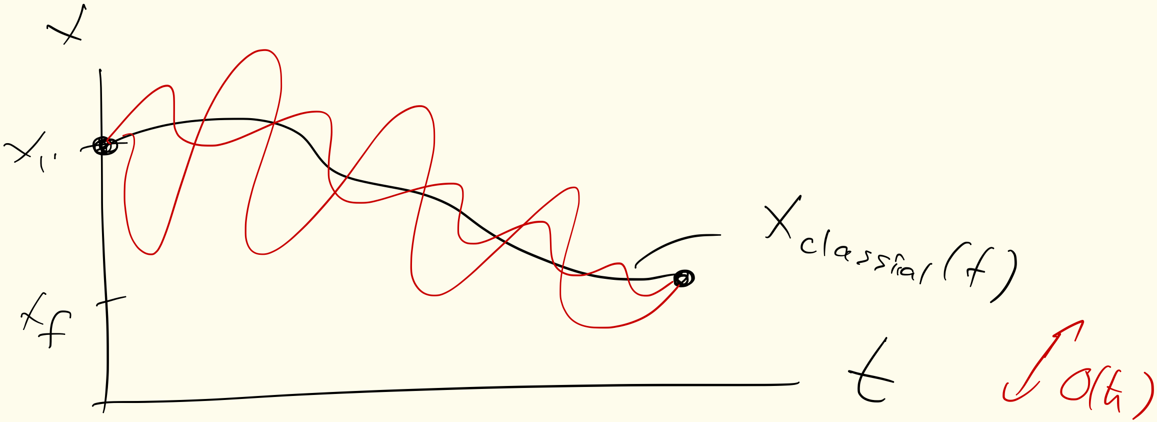
Wick-rotation of time

$$\underline{G = \int Dx \, e^{\frac{i}{\hbar} S}}$$



$$\frac{\delta S}{\delta x} = 0$$

$$x_{classical}(t)$$



$$S[x] = S[\bar{x} + y]$$

$$= S[\bar{x}] + \frac{1}{2}\int dt\, dt' \; y(t')\, A(t,t')\, y(t) + \cdots$$

$$A = \frac{\delta^2 S}{\delta x(t)\, \delta x(t')}$$

$$G = \int Dx \; e^{-\frac{1}{\hbar} S} = \sum_i \underline{e^{-\frac{1}{\hbar} S[\bar{x}_i]}} \; \underline{\underline{\left(\det\left(\frac{A_i}{2\pi}\right)\right)^{-\frac{1}{2}}}}$$

$$G = \int \mathcal{D}x \; e^{\frac{i}{\hbar}\int_0^t dt' \, \frac{m}{2}\left(\frac{\partial x}{\partial t'}\right)^2}$$

$$= \langle x_f | \, e^{-\frac{i}{\hbar}\frac{\hat{p}^2}{2m}t} \, | x_i \rangle$$

$$\hat{1} = \frac{1}{2\pi}\int dp \; |p\rangle\langle p|$$

$$= \frac{1}{2\pi}\int dp \; e^{-i/\hbar\left(\frac{p^2}{2m}t + p(x_i - x_f)\right)}$$

$$= \sqrt{\frac{m}{2\pi\hbar \, i t}} \; e^{i(x_f - x_i)^2 m/2\hbar t}$$