4 Functional Integrals

$$e^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n$$

$$|\phi\rangle = \sum_{n_1, n_2 \dots} C_{n_1 n_2} |n_1, n_2, \dots\rangle$$

$$\frac{(a_1^\dagger)^{n_1}}{\sqrt{n_1!}} \frac{(a_2^\dagger)^{n_2}}{\sqrt{n_2!}} \dots |0\rangle$$

Start with bosons:

Q: can we diagonalize a?

A: yes! $\longrightarrow$ coherent states $|\phi\rangle = \exp\left\{\sum_i \phi_i \overset{\text{scalar}}{\downarrow} a_i^\dagger\right\}|0\rangle$

check: $a_i|\phi\rangle = a_i \exp\left\{\sum_j \phi_j a_j^\dagger\right\}|0\rangle = \exp\left\{\sum_{j\neq i} \phi_j a_j^\dagger\right\} a_i e^{\phi_i a_i^\dagger}|0\rangle$

$= \exp\{\dots\} a_i \sum_{n=0}^{\infty} \frac{1}{n!} \phi_i^n (a_i^\dagger)^n |0\rangle = \exp\{\dots\} \sum_{n=1}^{\infty} \frac{1}{n!} \phi_i^n n (a_i^\dagger)^{n-1} |0\rangle$

$= \exp\left\{\sum_{j\neq i} \dots\right\} \sum_{n=0}^{\infty} \frac{1}{n!} \phi_i^{n+1} (a_i^\dagger)^n |0\rangle$ To check: $a(a^\dagger)^n|0\rangle$

$= \phi_i \exp\left\{\sum_{j\neq i} \dots\right\} \exp(\phi_i a_i^\dagger)|0\rangle = \phi_i \exp\left\{\sum_j \phi_j a_j^\dagger\right\}|0\rangle = \phi_i |\phi\rangle$ $= n(a^\dagger)^{n-1}|0\rangle$

$$\underline{a\, a^\dagger} \underbrace{\ldots a^\dagger}_{n} |0\rangle \qquad a a^\dagger = a^\dagger a + 1$$

THUS: $$\boxed{a_i |\phi\rangle = \phi_i |\phi\rangle}$$

Dual coherent ~~states~~: taking Hermitian conjugate,

$$\langle\phi| a_i^\dagger = \bar{\phi}_i \langle\phi| \qquad \text{where } \bar{\phi}_i \text{ is here } \phi_i^*$$

What about $a^\dagger$? Calculate it:

$$a_i^\dagger |\phi\rangle = a_i^\dagger \exp\Big\{\sum_j \phi_j a_j^\dagger\Big\}|0\rangle = \exp\Big\{\sum_{j\neq i} \ldots\Big\}\, a_i^\dagger e^{\phi_i a_i^\dagger} |0\rangle$$

$$= \exp\{\ldots\} \sum_{n=0}^{\infty} \frac{1}{n!} \phi_i^n (a_i^\dagger)^{n+1} |0\rangle = \exp\Big\{\sum_{j\neq i} \ldots\Big\}\, \partial_{\phi_i} \sum_{n=0}^{\infty} \frac{1}{n!} \phi_i^n a_i^{\dagger n} |0\rangle$$

so $$\boxed{a_i^\dagger |\phi\rangle = \partial_{\phi_i} |\phi\rangle}$$

Test: $[a_i, a_j^\dagger]|\phi\rangle = a_i a_j^\dagger|\phi\rangle - a_j^\dagger a_i|\phi\rangle$

$$= a_i \partial_{\phi_j}|\phi\rangle - a_j^\dagger \phi_i|\phi\rangle = \partial_{\phi_j} a_i|\phi\rangle - \phi_i a_j^\dagger|\phi\rangle$$

$$= \partial_{\phi_j}\{\phi_i|\phi\rangle\} - \phi_i \partial_{\phi_j}|\phi\rangle = \delta_{ij}|\phi\rangle$$

Overlap of coherent states:

$$\langle\theta|\phi\rangle = \langle 0|\exp\Big\{\sum_i \bar{\theta}_i a_i\Big\}|\phi\rangle = \langle 0|\exp\Big\{\sum_i \bar{\theta}_i \phi_i\Big\}|\phi\rangle$$

$$= \exp\Big\{\sum_i \bar{\theta}_i \phi_i\Big\}\underbrace{\langle 0|\phi\rangle}_{1} = \exp\sum_i \bar{\theta}_i\phi_i$$

$$\searrow \sum_{n=0}^{\infty}\frac{1}{n!}\Big(\sum_i \phi_i a_i^\dagger\Big)^n|0\rangle = |0\rangle + (\text{at least one } a^\dagger \ldots)$$

$$\boxed{\langle\theta|\phi\rangle = \exp\Big\{\sum_i \bar{\theta}_i\phi_i\Big\}}$$

Completeness relation:

Usually, for e.g. a finite-dim^{nl} Hilbert space, you would write $\mathbb{1} = \sum_\alpha |\alpha\rangle\langle\alpha|$

eigenbasis

For coherent states: need to "rescale" each term.

Statement:
$$\mathbb{1}_F = \int \prod_i \frac{d\bar\phi_i \, d\phi_i}{\pi} \, e^{-\sum_i \bar\phi_i \phi_i} \, |\phi\rangle\langle\phi|$$

Fock space

(note: the norm of a coherent state is
$$\langle\phi|\phi\rangle = \exp\left\{\sum_i \bar\phi_i \phi_i\right\}$$
)

Proof: show that $a_i \mathbb{1}_F = \mathbb{1}_F a_i$ (see notes)

Coherent states for fermions.

Seek states $|\eta\rangle$ such that $a_i|\eta\rangle = \eta_i|\eta\rangle$

fermionic, anticommute

Strangely, we need also the eigenvalues to anticommute!

For algebraic consistency, need $\eta_i\eta_j = -\eta_j\eta_i \quad \forall\, i,j$

Anticommuting numbers form a Grassmann algebra

Fun thing: $\eta_i^2 = 0$

What is the most general f^{ns} of a Grassmann var η?

$$f(\eta) = f_0 + f_1\eta$$

Define the differential operator as: $\partial_{\eta_i} \eta_j = \delta_{i,j}$

Also, $\partial_{\eta_1}(\eta_2 \eta_3 \ldots) = -\eta_2 \partial_{\eta_1}(\eta_3 \ldots)$

"derivative is also Grassmann" (anticommutes)

Define integration by the 2 rel^{n}s

$$\int d\eta_i = 0 \qquad \int d\eta_i \, \eta_i = 1 \quad \text{(literally)}$$

Cool consequence:

$$\int d\eta \, f(\eta) = \int d\eta \, (f_0 + f_1 \eta) = f_0 \underbrace{\int d\eta}_{0} + f_1 \underbrace{\int d\eta \, \eta}_{1} = f_1$$

$$\partial_\eta f(\eta) = \partial_\eta (f_0 + f_1 \eta) = f_1$$

Coherent states: similar to the bosonic case (simpler, actually)

$$|\eta\rangle = \exp\Big\{-\sum_i \eta_i a_i^\dagger\Big\}|0\rangle$$

same

Check: $$a_i|\eta\rangle = a_i \exp\Big\{+\sum_j a_j^\dagger \eta_j\Big\}|0\rangle$$

$$= \exp\Big\{-\sum_{j\neq i}\eta_j a_j^\dagger\Big\}\, a_i\, \underbrace{\exp\{-\eta_i a_i^\dagger\}}_{1-\eta_i a_i^\dagger + \frac{1}{2}\cancel{(\eta_i a_i^\dagger)^2}_{0} + \dots}|0\rangle$$

$$= \exp\Big\{-\sum_{j\neq i}\eta_j a_j^\dagger\Big\}\underbrace{\left[a_i - a_i\eta_i a_i^\dagger\right]|0\rangle}_{=\cancel{a_i|0\rangle}_{0} + \eta_i a_i a_i^\dagger|0\rangle = \eta_i|0\rangle}$$

$$= \exp\Big\{-\sum_{j\neq i}\eta_j a_j^\dagger\Big\}\eta_i|0\rangle$$

$$= \exp\Big\{-\sum_{j\neq i}\eta_j a_j^\dagger\Big\}\eta_i\underbrace{(1-\eta_i a_i^\dagger)|0\rangle}_{\exp(-\eta_i a_i^\dagger)|0\rangle} = \eta_i \exp\Big\{-\sum_j \eta_j a_j^\dagger\Big\}|0\rangle$$

$$= \eta_i|\eta\rangle$$

Adjoint: $\langle \eta | = \langle 0 | \exp\left\{-\sum_i a_i \bar{\eta}_i\right\} = \langle 0 | \exp\left\{\sum_i \bar{\eta}_i a_i\right\}$

(N.B.: $\{\eta_i, a_j\} = 0$)

$= \eta_i a_j + a_j \eta_i$

Gaussian:

$$\int d\bar{\eta}\, d\eta\, e^{-c\bar{\eta}\eta} = \int d\bar{\eta}\, d\eta \left\{1 - c\bar{\eta}\eta\right\}$$

$$= \underbrace{\cancel{\int d\bar{\eta}\, d\eta}}_{0} - c\int d\bar{\eta}\, d\eta\, \bar{\eta}\eta = +c\int d\bar{\eta} \underbrace{\left[\int d\eta\, \eta\right]}_{=1} \bar{\eta}$$

$$= c\int d\bar{\eta}\, \bar{\eta} = c$$

Resolution of identity:

$$\mathbb{1} = \int \prod_i d\bar{\eta}_i\, d\eta_i\, e^{-\sum_i \bar{\eta}_i \eta_i} |\eta\rangle\langle\eta|$$

(N.B.: $\bar{\eta}$ is <u>not</u> a "complex conjugate" of η: it's just <u>another</u> Grassmann)