Pauli

$$H = \sum_{\alpha b} a^\dagger_{\alpha b}\left[\frac{p_\alpha^2}{2m} - \frac{\mu_0 B}{2}\sigma^z_{bb}\right] a_{\alpha b}$$

$b \in \uparrow, \downarrow$

$$\sigma^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$= \sum_\alpha a^\dagger_{\alpha\uparrow}\left[\frac{p_\alpha^2}{2m} - \frac{\mu_0 B}{2}\right] a^\dagger_{\alpha\uparrow}$$

$$+ \sum_\alpha a^\dagger_{\alpha\downarrow}\left[\frac{p_\alpha^2}{2m} - \frac{\mu_0 B}{2}(-1)\right] a_{\alpha\downarrow} = \sum_\alpha \sum_{\sigma=\pm} \varepsilon_{\alpha\sigma}\, a^\dagger_{\alpha\sigma} a_{\alpha\sigma}$$

$$\varepsilon_{\alpha\sigma} = \frac{p^2}{2m} - \frac{\mu_0 B}{2}\sigma$$

CSFFI:

$$Z = \int \mathcal{D}(\bar\psi, \psi)\, e^{-S[\bar\psi,\psi]}$$

$$\xi_\alpha = \frac{p_\alpha^2}{2m} - \mu$$

$$S[\bar\psi,\psi] = \sum_\alpha \sum_\sigma \sum_n \bar\psi_{\alpha\sigma n}\left[-i\omega_n + \xi_\alpha - \frac{\mu_0 B}{2}\sigma^z_{\sigma\sigma}\right]\psi_{\alpha\sigma n}$$

$$Z = \prod_{\alpha\sigma n} \int \mathcal{D}(\bar\psi_{\alpha\sigma n}, \psi_{\alpha\sigma n}) \exp\left\{-\bar\psi_{\alpha\sigma n}\left[-i\omega_n + \xi_\alpha - \frac{\mu_0 B}{2}\sigma\right]\psi_{\alpha\sigma n}\right\}$$ (where we take $\sigma = \pm$)

Integral: remember that $\int \mathcal{D}(\bar\psi,\psi)\, e^{-\bar\psi c \psi} = (\beta c)^{-\zeta}$

For fermions: Grassmann integral:

$$\int \underbrace{\mathcal{D}(\bar\psi,\psi)}_{\beta\, d\bar\psi\, d\psi} e^{-\bar\psi c \psi} = \beta \int d\bar\psi\, d\psi\, \underbrace{e^{-\bar\psi c\psi}}_{1-\bar\psi c\psi} = -\beta c \int d\bar\psi\, d\psi\, \underbrace{\bar\psi \psi}_{-\psi\bar\psi}$$

$\mathbb{C}$

$$= \beta c \int d\bar\psi \underbrace{\int d\psi\, \psi}_{1} \bar\psi = \beta c$$

So $$Z = \prod_{\alpha\sigma n} \left[\beta\left(-i\omega_n + \xi_\alpha - \frac{\mu_0 B}{2}\sigma\right)\right] = \prod_{\alpha n}\left\{\beta^2\left[(-i\omega_n+\xi_\alpha)^2 - \frac{\mu_0^2 B^2}{4}\right]\right\}$$

$$F = -T\sum_{\alpha n} \ln\left[\beta^2\left[(-i\omega_n + \xi_\alpha)^2 - \frac{\mu_0^2 B^2}{4}\right]\right]$$

Susceptibility: $\chi = \partial_B \underset{\substack{\uparrow \\ \text{magnetiz}^{\text{n}}}}{M}\Big|_{B=0} = -\partial_B^2 F\Big|_{B=0} = -\frac{\mu_0^2 T}{2}\sum_{\alpha n} \frac{1}{(-i\omega_n+\xi_\alpha)^2}$

Perform Matsubara sum using useful formula ...

Electron-phonon coupling (4.5.7)

$$H_{ph} = \sum_{\underset{\sim}{q}} \sum_{j} \omega_q \, a^{\dagger}_{\underset{\sim}{q}j} a_{\underset{\sim}{q}j}$$

Lattice vibrations induce a local +ve charge given by polarisation operator through $\rho_{ind} = \nabla \cdot \underset{\sim}{P}$

$P \sim u \sim a^{\dagger} + a$ "displacement" of ions

Coupling between e^-s & phonons:

$a + a^{\dagger} \sim \phi + \bar{\phi}$

$$H_{el-ph} = \gamma \int d^3 r \; \hat{n}(\underset{\sim}{r}) \, \nabla \cdot u(\underset{\sim}{r})$$

density of e^-s $\quad f^{\dagger} f$

$\downarrow$

$\bar{\psi}\psi$

Partition function:
$$Z = \int \mathcal{D}[\bar\psi, \psi] \int \mathcal{D}[\bar\phi, \phi]\, e^{-S_{el} - S_{ph} - S_{el-ph}}$$

$\int \mathcal{D}[\bar\psi,\psi]$: for e^-

$\int \mathcal{D}[\bar\phi,\phi]$: for phonons (bosons)

$$S_{ph} = \sum \bar\phi\,(-i\omega_n + \xi)\,\phi$$

$$S_{el-ph} = \sum \bar\psi\psi\,(\phi + \bar\phi)$$

$$\int \mathcal{D}[\bar\phi,\phi]\, e^{-\bar\phi[\,]\phi - [\phi+\bar\phi]\bar\psi\psi} \sim e^{+(\bar\psi\psi)^2}$$

Phonon integral: Gaussian $\rightarrow$ integrate out the phonons!

$\rightarrow$ generates effective interaction between fermions

$$Z = \int \mathcal{D}[\bar\psi,\psi]\, e^{-S_{eff}[\bar\psi,\psi]}$$

$$S_{eff} = S_{el} - \gamma^2 \sum (\bar\psi\psi)(\bar\psi\psi)$$

Perturbation theory

Toy integral: $$I(g) = \int_{-\infty}^{\infty} \frac{dx}{\sqrt{2\pi}}\, e^{-\frac{x^2}{2} - gx^4}$$

$$e^{-gx^4} = \sum_{n=0}^{\infty} \frac{(-g)^n}{n!} x^{4n}$$

$g=0$: OK

For small g, expand $I(g) = \sum_{n=0}^{\infty} g^n I_n$

Coefficients explicitly given by $$g^n I_n = \frac{(-g)^n}{n!} \underbrace{\int_{-\infty}^{\infty} \frac{dx}{\sqrt{2\pi}}\, e^{-\frac{x^2}{2}} x^{4n}}$$

$$\equiv \frac{(-g)^n}{n!} \langle x^{4n} \rangle$$ OK, see notes

Exact calculation: $\langle x^{4n} \rangle = (4n-1)!!$
$= (4n-1)(4n-3)\ldots$

How to get this: consider $$I_a^0 = \int_{-\infty}^{\infty} \frac{dx}{\sqrt{2\pi}}\, e^{-ax^2/2} = \frac{1}{\sqrt{a}}$$

Take $\partial_a I_a^0$: $\sim \langle x^2 \rangle$

Let's estimate: $g^n I_n = (-g)^n \dfrac{(4n-1)!!}{n!}$

$N! = N^N e^{-N}(1+\dots)$

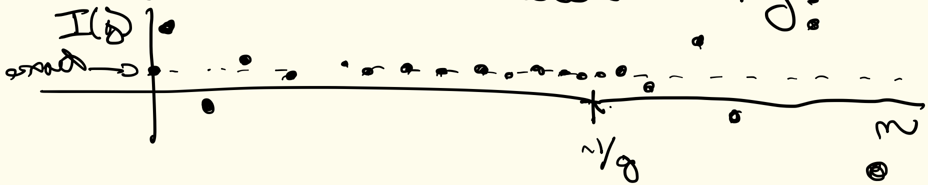
Use Stirling: $\ln N! = N \ln N - N + \dots$ for N large

$$\to \quad g^n I_n \sim (-g)^n \frac{\left[(4n)^{4n} e^{-4n}\right]^{1/2}}{n^n e^{-n}} \sim \left(-\underset{\substack{\uparrow \\ \text{nr of } O(1)}}{(\#)}\,\frac{gn}{e}\right)^n$$

Perturbation theory <u>cannot</u> converge!

$\forall g$, $\exists n$ s.t. $n \sim 1/g$, at which the series starts to diverge.

Best strategy: truncate series at order $\sim 1/g$.



Simple example: ϕ^4 theory

$$Z = \int \mathcal{D}\phi \, e^{-S[\phi]}$$

$$S[\phi] = \int d^d x \left[\frac{1}{2} (\underbrace{\partial\phi)^2}_{\partial_\mu\phi\,\partial^\mu\phi} + \frac{m^2}{2}\phi^2 + g\phi^4 \right]$$

Case $g=0$ is Gaussian $\rightarrow$ exactly solvable.

Standard notation: $$\langle \ldots \rangle \equiv \frac{\int \mathcal{D}\phi \, (\ldots)\, e^{-S[\phi]}}{\int \mathcal{D}\phi \, e^{-S[\phi]}}$$

Also: $$\langle \ldots \rangle_0 = \frac{\int \mathcal{D}\phi \, (\ldots)\, e^{-S_0[\phi]}}{\int \mathcal{D}\phi \, e^{-S_0[\phi]}} \qquad S_0 = S\Big|_{g=0}$$

Correlation fns:

$$\langle \phi(\underset{\sim}{x}_1)\phi(\underset{\sim}{x}_2)\ldots\phi(\underset{\sim}{x}_n)\rangle \equiv C_n(\underset{\sim}{x}_1, \ldots, \underset{\sim}{x}_n)$$

Most important: 2-pt f^{n} $C_2(\underset{\sim}{x}_1, \underset{\sim}{x}_2) = \mathcal{G}(\underset{\sim}{x}_1 - \underset{\sim}{x}_2)$

f^{n} of position difference due to translational inv.

Basic calculation: 2-pt f^{n} of Gaussian model

Introduce FT: $\phi(\underset{\sim}{x}) = \frac{1}{L^{d/2}} \sum_{\underset{\sim}{p}} e^{-i \underset{\sim}{p} \cdot \underset{\sim}{x}} \phi_{\underset{\sim}{p}}$ d: dim^{n} of space

Action: becomes $S_0[\phi] = \sum_{\underset{\sim}{p}} \frac{1}{2} \phi_{\underset{\sim}{p}} (p^2 + m^2) \phi_{-\underset{\sim}{p}}$ $p^2 \equiv \underset{\sim}{p} \cdot \underset{\sim}{p}$

So: $\langle \phi(\underset{\sim}{x}) \phi(0) \rangle_0 = \mathcal{G}_0(\underset{\sim}{x}) = \frac{1}{L^d} \sum_{\underset{\sim}{p}, \underset{\sim}{p}'} e^{-i \underset{\sim}{p} \cdot \underset{\sim}{x}} \langle \phi_{\underset{\sim}{p}} \phi_{\underset{\sim}{p}'} \rangle_0$

From Gaussian integrals: $\langle \phi_{\underset{\sim}{p}} \phi_{\underset{\sim}{p}'} \rangle_0 = \delta_{\underset{\sim}{p} + \underset{\sim}{p}', 0} \frac{1}{p^2 + m^2}$

so $\mathcal{G}_0(\underset{\sim}{x}) = \frac{1}{L^d} \sum_{\underset{\sim}{p}} e^{-i \underset{\sim}{p} \cdot \underset{\sim}{x}} \frac{1}{p^2 + m^2} \xrightarrow[L \to \infty]{} \int \frac{d^d p}{(2\pi)^d} \frac{e^{-i \underset{\sim}{p} \cdot \underset{\sim}{x}}}{p^2 + m^2}$

This is called a Green's function, because it satisfies

$$\left[-\partial_{\underset{\sim}{r}}^2 + m^2 \right] G(\underset{\sim}{r} - \underset{\sim}{r}') = \delta(\underset{\sim}{r} - \underset{\sim}{r}')$$

Perturbation theory at low orders

$$S = S_0 + S_{int}$$

$$S_{int} \equiv g \int d^d y \, \phi^4(y)$$

Generic object in PT

$$\langle X[\phi] \rangle \simeq \frac{\sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \langle X[\phi] S_{int}^n \rangle_0}{\sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \langle S_{int}^n \rangle_0}$$

Ex.:

$$\langle \phi(x)\phi(x') \rangle = \frac{\langle \phi(x)\phi(x') \rangle_0 - g\int d^d y \langle \phi(x)\phi^4(y)\phi(x') \rangle_0 + \frac{g^2}{2}\int dy_1 dy_2 \langle \phi(x)\phi^4(y_1)\phi^4(y_2)\phi(x') \rangle_0 + \dots}{1 - g\int d^d y \langle \phi^4(y) \rangle_0 + \frac{g^2}{2}\int dy_1 dy_2 \langle \phi^4(y_1)\phi^4(y_2) \rangle_0 + \dots}$$

Pictorial tricks: use drawings to represent individual terms in series expansion.

Basic elements:

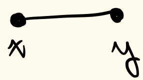
One comment before this: calculating Gaussian averages

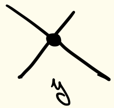
Look e.g. at $\langle \phi(\underline{x})\phi^4(y)\phi(x')\rangle_0$

Wick's theorem: average of a product of fields, is the product of pairwise averages, summed over all pairings.

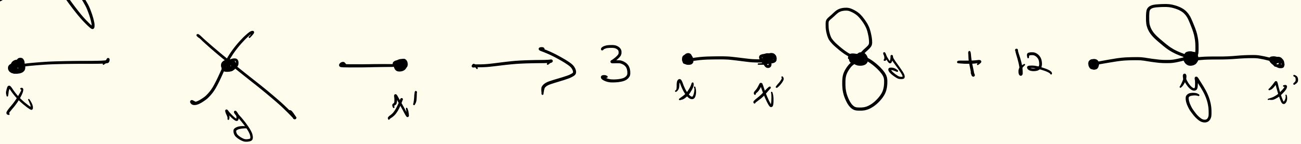
$$\langle \phi_1\phi_2\phi_3\phi_4\rangle_0 = \langle\phi_1\phi_2\rangle_0\langle\phi_3\phi_4\rangle_0 + \langle\phi_1\phi_3\rangle_0\langle\phi_2\phi_4\rangle_0 + \langle\phi_1\phi_4\rangle_0\langle\phi_2\phi_3\rangle_0$$

So
$$\langle\phi(x)\phi^4(y)\phi(x')\rangle_0 = 3\langle\phi(x)\phi(x')\rangle_0\left[\langle\phi(y)\phi(y)\rangle_0\right]^2 + 12\langle\phi(x)\phi(y)\rangle_0\langle\phi(y)\phi(y)\rangle_0\langle\phi(y)\phi(x')\rangle_0$$

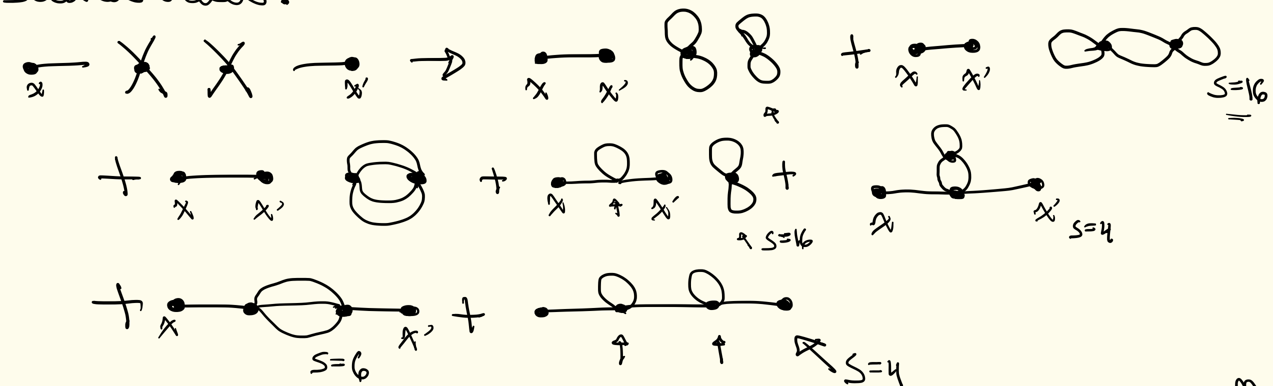
Represent a $G_0(x-y) = \langle \phi(x)\phi(y)\rangle_0$ as 

" interaction term $g\phi^4(y)$ as 

To first order:



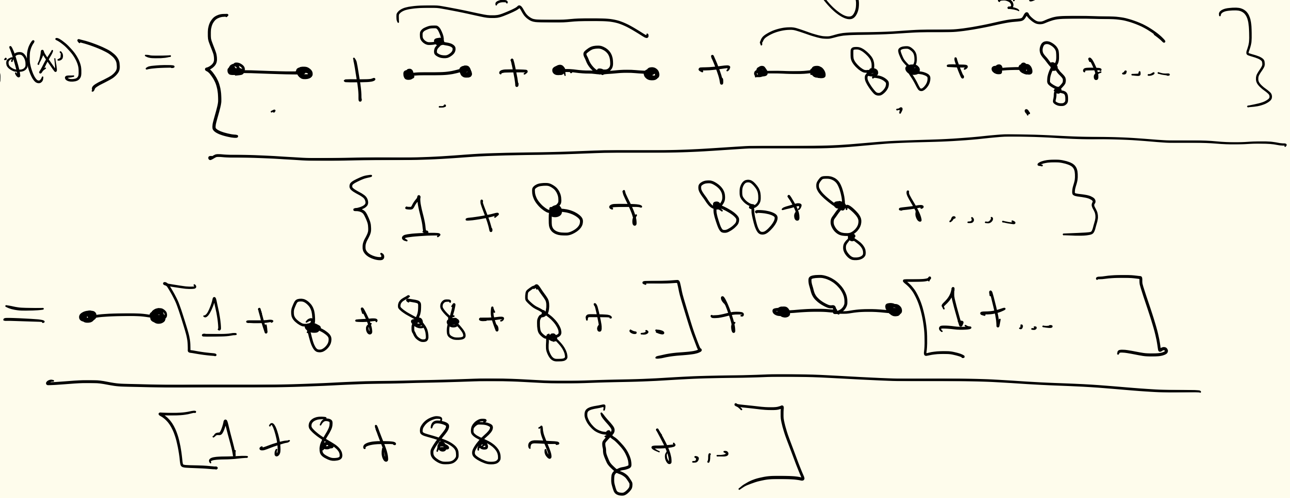
Second order:



Coefficients: use "symmetry factor"   Coefficient here: $(4!)^n/S$ (here $n=2$)

Linked cluster theorem: only connected diagrams appear in series expansion for correlator

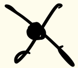
$\langle \phi(x)\phi(x') \rangle =$



+ only connected terms

Feynman rules:

To compute the order n contribution to a given correlator,

- for each operator $\phi(x_i)$, draw  x_i
- draw n copies of interaction vertex  $y_j \quad j=1,\dots,n$
- draw all topologically distinct connected diagrams by joining lines pairwise
- integrate all y_j
- put prefactor on each term using division by symmetry factor (times $(4!)^n$ in our convention)