Lecture 9a

$V \propto r_0^3$

$\Delta p_x \propto \frac{\hbar}{r_0}$

$E_{kin} \propto \frac{\hbar^2}{m r_0^2}$

$E_{int} \propto \frac{e^2}{r_0}$

$E_{int} / E_{kin} = 1$

$$\frac{e^2}{r_0} \frac{m r_0^2}{\hbar^2} = \frac{r_0}{\hbar^2 / m e^2} = \frac{r_0}{a_0}$$

$$\equiv r_s$$

$r_s \ll 1$ $\quad E_{kin} \gg E_{int}$ $\quad$ fermi gas

$r_s \gg 1$ $\quad E_{int} \gg E_{kin}$ $\quad$ new order

$\boxed{r_s = \mathcal{O}(1)} \longrightarrow$ Landau liquid.

weakly interacting electron gas.

$$F = -T \ln Z$$

$p = (\vec{p}, i\omega_n)$

$$Z = \int D[\psi, \bar\psi]\, e^{-S[\psi,\bar\psi]}$$

$$S = S_0 + S_{int} \longleftarrow S_0 = \sum_{p,\sigma} \bar\psi_{p,\sigma}\left(-i\omega_n - \mu + \frac{|p|^2}{2m}\right)\psi_{p,\sigma}$$

$$S_{int} = \frac{T}{2L^3} \sum_{\substack{p p' q \\ \sigma \sigma'}} \bar\psi_{p+q,\sigma}\, \psi_{p,\sigma}\, V(q)\, \bar\psi_{p'-q,\sigma'}\, \psi_{p',\sigma'}$$

"jellium"

$$V(q) = \begin{cases} \dfrac{e^2}{q^2} & q \neq 0 \\ 0 & q = 0 \end{cases} \qquad V(r) = \frac{e^2}{|r|}$$

$q=0 \longleftarrow$ charge neutrality

$$F_0 = -T \ln \underline{\int D[\bar{\psi},\psi]\, e^{-S_0[\bar{\psi},\psi]}}$$

$$= -T \ln \prod_{\vec{p},n} \left(-i\omega_n - \mu + \frac{|\vec{p}|^2}{2m}\right)^{-1}$$

$$= -T \sum_{\vec{p},n} \ln \left(-i\omega_n - \mu + \frac{|p|^2}{2m}\right)^{-1}$$

$$= -T \sum_{\vec{p}} \ln \left(1 + e^{-\beta\left(\frac{|p|^2}{2m} - \mu\right)}\right)$$

$\lim_{T \to 0}$: $\quad F_0 \to \sum_{|p|<p_F} \left(\frac{|p|^2}{2m} - \mu\right)$

$$\frac{F_0}{N} = \frac{\int_0^{p_F} dp\, p^2 \left(\frac{p^2}{2m}\right)}{\int_0^{p_F} dp\, p^2 \,(1)} = \frac{1}{2m} \frac{\frac{1}{5} p_F^5}{\frac{1}{3} p_F^5} = \frac{3}{10} p_F^2 / m$$

$$p_F = \sqrt{2m\mu} \longrightarrow \frac{F_0}{N} = \frac{3}{5}\mu$$

$$\mu = \frac{\hbar^2}{2m}\left(3\pi^2 \underline{\frac{N}{V}}\right)^{2/3} \qquad \frac{V}{N} = r_0^3 \cdot \frac{4\pi}{3}$$

$$\Longrightarrow \boxed{\frac{F_0}{N} \propto \frac{1}{r_s^2}}$$

$$Z = \int D[\bar{\psi}, \psi] \, e^{-S_0 - S_{int}}$$

$$= \int D[\bar{\psi}, \psi] \, e^{-S_0} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} S_{int}^n$$

$$= Z_0 \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \langle S_{int}^n \rangle_0$$

$$\langle \ldots \rangle_0 = \frac{\int D[\bar{\psi}, \psi] \, e^{-S_0} \ldots\ldots}{\int D[\bar{\psi}, \psi] \, e^{-S_0}} = \frac{\int D \, e^{-S_0} \ldots\ldots}{Z_0}$$

$$F = -T \ln Z = \underbrace{-T \ln Z_0}_{F_0} - T \ln \left[\sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \langle S_{int}^n \rangle_0 \right]$$

$$\left[1 + \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} \langle S_{int}^n \rangle_0 \right]$$

$$\ln(1+x) = \sum_{m=1}^{\infty} \frac{(-1)^{m+1}}{m} x^m$$

$$\Rightarrow F - F_0 = -T \sum_{m=1}^{\infty} \frac{(-1)^{m+1}}{m} \left[\sum_{n=1}^{\infty} \frac{(-1)^n}{n!} \langle S_{int}^n \rangle_0 \right]^m$$

$$= -T \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} \langle S_{int}^n \rangle_0$$

(connected!)

$$m=2 \; \propto \langle S_{int}^n \rangle_0 \langle S_{int}^{n'} \rangle_0$$

Lecture 9b

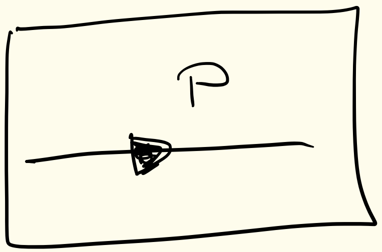
$$F - F_0 = -T \sum_n \frac{(-1)^n}{n!} \langle S_{int}^n \rangle_0$$

1) fermion propagator

$$\frac{\int d\psi_p \, \bar{\psi}_p \psi_p \int\int\int\int \cdots}{\int d\psi_p \int\int\int\int \cdots} \qquad \int D[\bar{\psi},\psi] = \int d\psi_{p_1} d\psi_{p_2} d\psi_{p_3}$$

$$\mathcal{G}_0(p) = \langle \bar{\psi}_{p\sigma} \psi_{p\sigma} \rangle_0$$

$$= \frac{\int D[\bar{\psi},\psi] \, e^{-S_0} \, \bar{\psi}_{p\sigma} \psi_{p\sigma}}{\int D[\bar{\psi},\psi] \, e^{-S_0}}$$

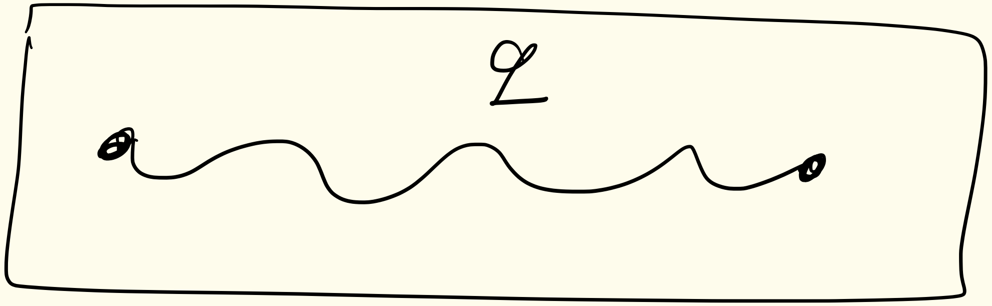


$$= \frac{\int d(\bar{\psi}_{p\sigma}, \psi_{p\sigma}) \, e^{-\bar{\psi}_{p\sigma}\left[-i\omega_n + \frac{|p|^2}{2m} - \mu\right]\psi_{p\sigma}} \, \bar{\psi}_{p\sigma}\psi_{p\sigma}}{\int d(\bar{\psi}_{p\sigma}, \psi_{p\sigma}) \, e^{-S_0}}$$

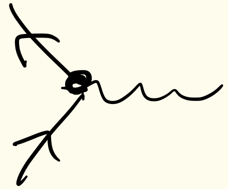
$$= \left(i\omega_n + \mu - |p|^2/2m\right)^{-1}$$

2) interaction Vertex

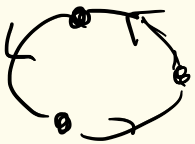
$V(q)$



3) Feynman <u>rules</u>

* conserve 4-momentum at every 

* fermion <u>loops</u> get a $(-1)2$



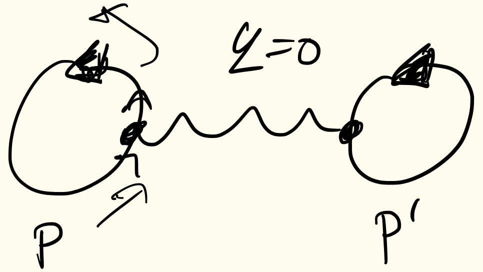
$$\langle 12\rangle\langle 23\rangle\langle 31\rangle$$

$$= -\langle 13\rangle$$

* sum over internal momenta & spin in ~~side~~ loops

$$F^{(1)} = -T\,(-1)^1 \langle S^1_{int}\rangle_0$$

$$= T\,\frac{T}{2L^3}\sum_{\substack{pp'q\\ \sigma\sigma'}} \langle \bar\psi_{p+q,\sigma}\,\psi_{p,\sigma}\,V(q)\,\bar\psi_{p'-q,\sigma'}\,\psi_{p',\sigma'}\rangle_0$$

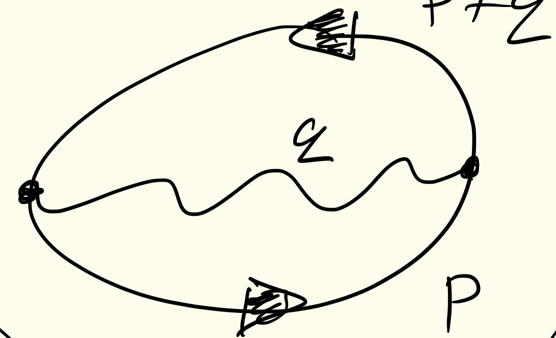


$$\frac{T^2}{2L^3}\sum_{\substack{pp'q\\ \sigma\sigma'}} \underbrace{\langle\bar\psi_{p+q,\sigma}\,\psi_{p,\sigma}\rangle_0}_{G_0(p)\,\delta_{q,0}}\,V(q)\,\langle\bar\psi_{p'-q,\sigma'}\,\psi_{p',\sigma'}\rangle_0$$

$$G_0(p')\,\delta_{q,0}$$

$$= 0$$

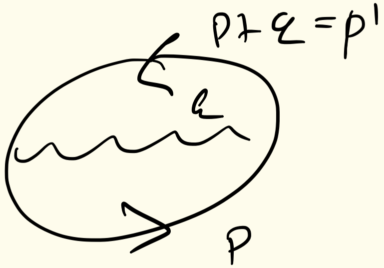
"Hartree"



$$\frac{T^2}{2L^3}\sum \langle\bar\psi_{p+q}\,\psi_{p'}\rangle_0\,V(q)\,\langle\psi_p\,\bar\psi_{p'-q}\rangle_0$$

$$G_0(p')\,\delta_{p',p+q}$$

"Fock"



$$F^{(1)} = -2\,\frac{T^2}{2L^3}\sum_{pp'} \mathcal{G}_0(p)\,\mathcal{G}_0(p')\,V(p-p')$$

$$= -\frac{1}{L^3}\sum_{\vec{p},\vec{p}'} n_F(\varepsilon_p)\,n_F(\varepsilon_{p'})\,\frac{e^2}{|\vec{p}-\vec{p}'|^2}$$

$$T\sum_n \mathcal{G}_0(p) = n_F(\varepsilon_p)$$

$T \to 0$

$$F^{(1)} \longrightarrow -\frac{1}{L^3}\sum_{|p|,|p'|<p_F} \frac{e^2}{(\vec{p}-\vec{p}')^2}$$

$$= -\frac{e^2 L^3}{(2\pi)^4}\,p_F^4$$

$$\propto -\frac{1}{r_s}$$

$$F_0 + F^{(1)} = \sum_{|p|<p_F} \left(\frac{|p|^2}{2m} - \frac{1}{L^3} \sum_{|p'|<p_F} \frac{e^2}{|\vec{p}-\vec{p}'|^2} \right)$$

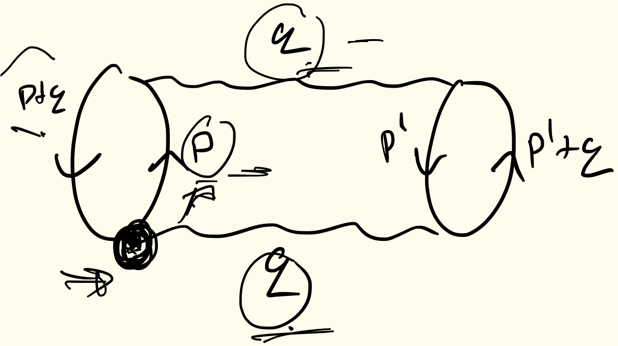
$$\equiv \sum_{|p|<p_F} \varepsilon^{(1)}(p)$$

$$\rho^{(1)}(\varepsilon) = \sum_p \delta(\varepsilon - \varepsilon^{1}(p))$$

$$\rho^{(1)}(\varepsilon)\Big|_{\varepsilon=\mu} = 0.$$

Lecture 9c

$$F^{(2)} = -T\,\frac{1}{2}\left(\frac{T}{2L^3}\right)^2 \left\langle \sum \bar{\psi}_{p+q}\,\psi_p\, V(q)\, \bar{\psi}_{p'-q}\,\psi_{p'}\; \bar{\psi}_{p''+q'}\,\psi_{p''}\, V(q')\, \bar{\psi}_{p'''-q'}\,\psi_{p'''} \right\rangle_0$$



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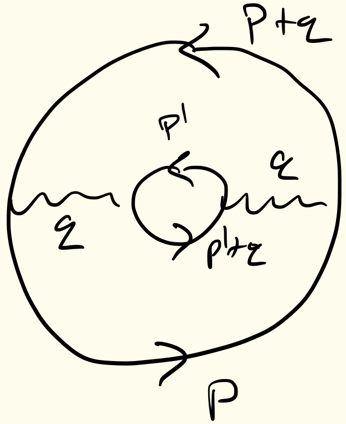
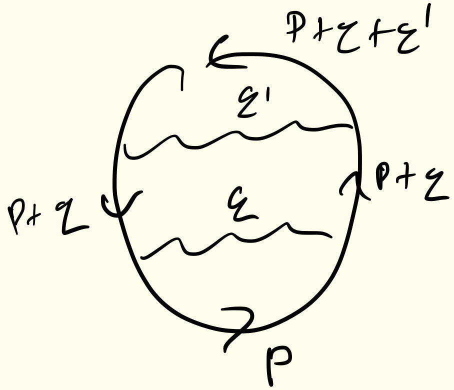


"Renormalization of the interaction"



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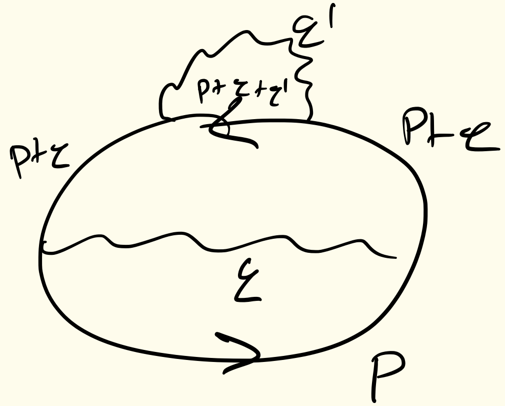
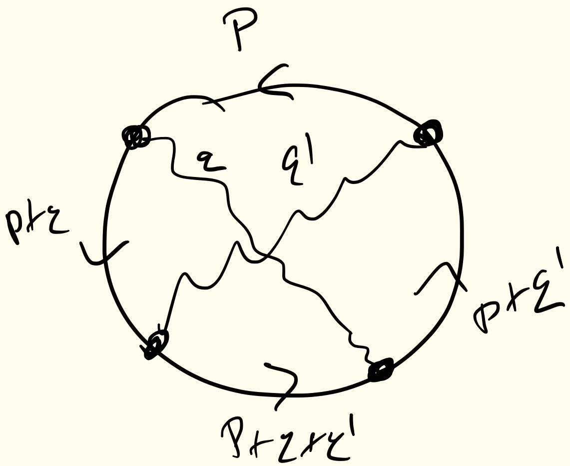


"Self-energy correction"



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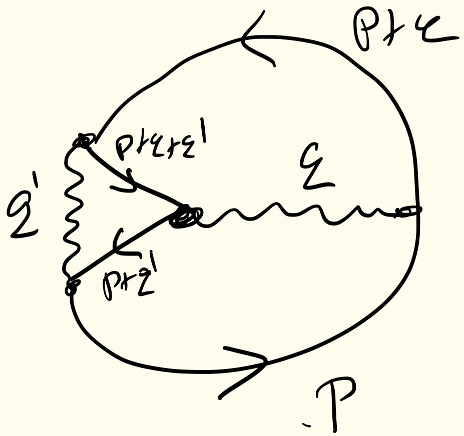
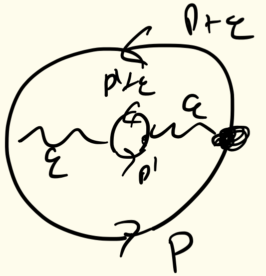


"Vertex corrections"



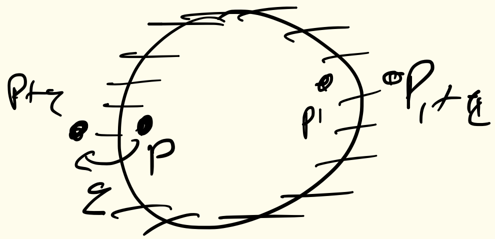
$$-T\,\frac{1}{2}\left(\frac{T}{2L^3}\right)^2 \cdot 2 \cdot (-2)^2 \sum_{pp'q} G_0(p)\, G_0(p+q) \cdot$$

$$G_0(p')\, G_0(p'+q) \cdot$$

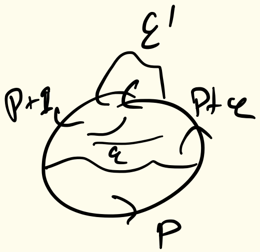
$$\underline{V(q)}\ \underline{V(q)}$$

$G_0(p)$

$p \sim p_F$



$$\propto A_F^2$$



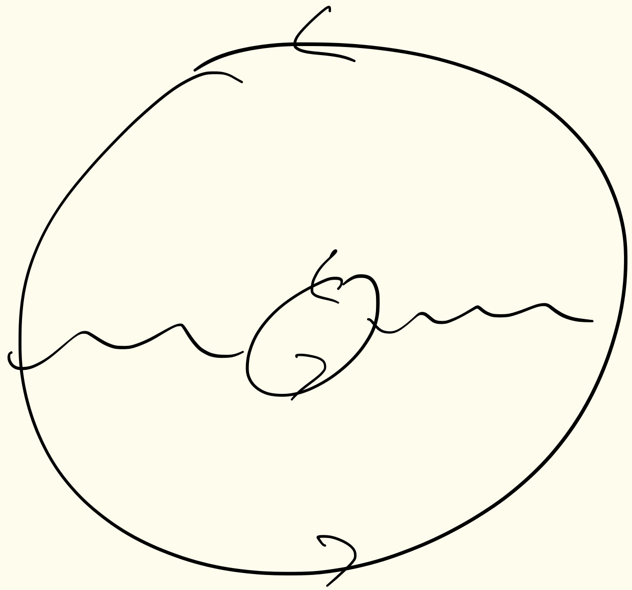
$$-T\,\frac{1}{2}\left(\frac{T}{2L^3}\right)^2 \cdot 4 \cdot (-2)^1 \sum_{pqq'} G_0(p)\, G_0(p+q)$$

$$G_0(p+q+q')\, G_0(p+q)$$

$$V(q)\ V(q')$$

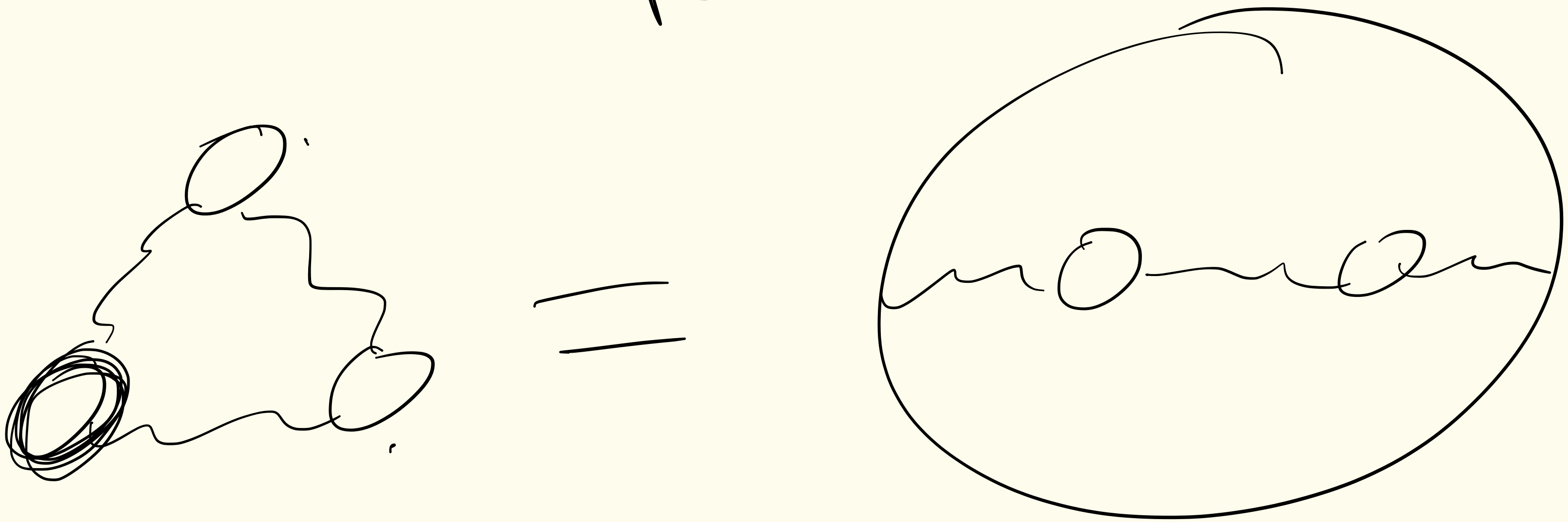
$p,\ p+q,\ p+q+q' \simeq p_F$

$$\propto A_F$$

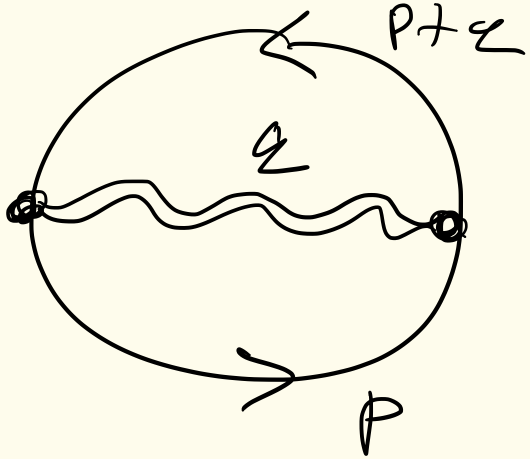


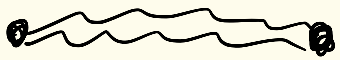
RPA

Random Phase approx.



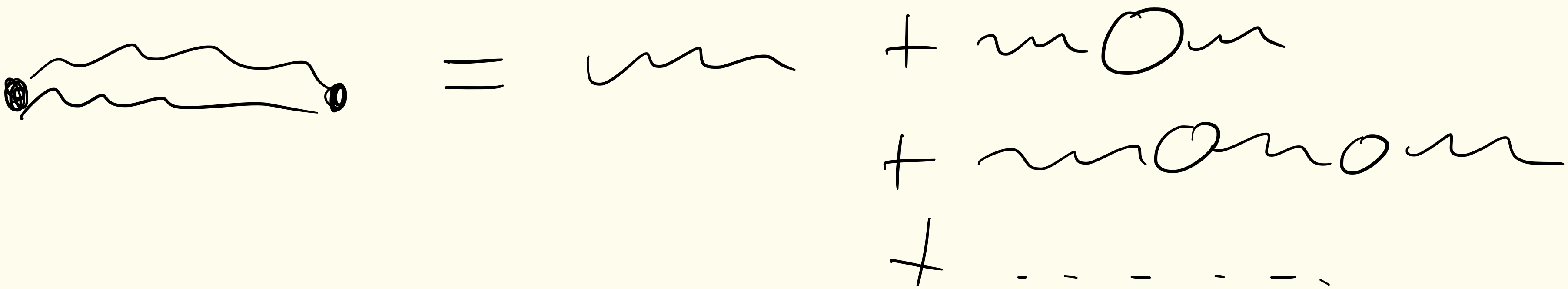
$F^{RPA} - F_0 =$

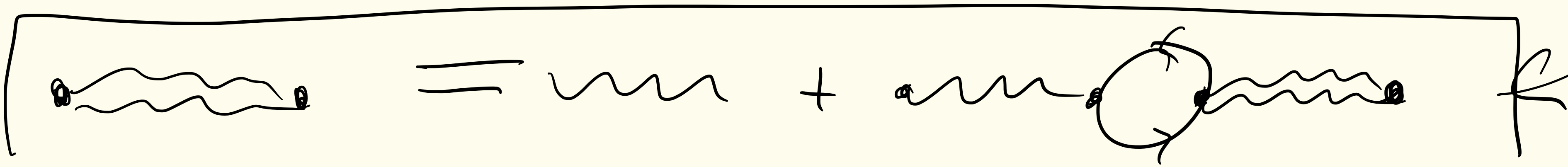


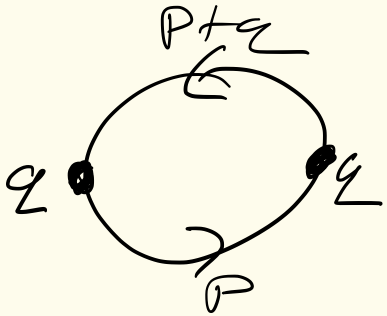


"Renormalized interaction"

Dyson equation







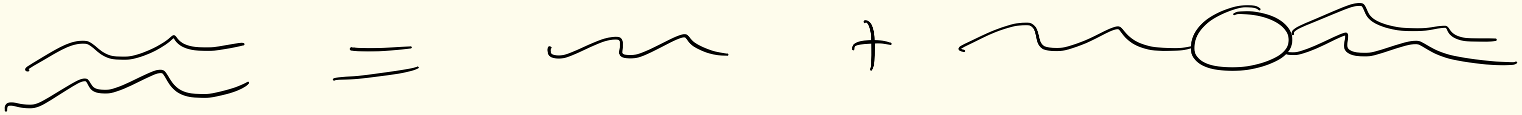
$$\equiv \Pi(q) \qquad \text{Polarization}$$

$$= \frac{2T}{L^3} \sum_{p} \mathcal{G}_0(p)\, \mathcal{G}_0(p+q)$$

$$= \frac{2}{L^3} \sum_{\vec{p}} \frac{n_F(\varepsilon_{p+q}) - n_F(\varepsilon_p)}{i\omega_n + \varepsilon_{p+q} - \varepsilon_p}$$

"Lindhard fct"

$\Pi(q)$



$$V^{RPA}(q) = V(q) + V(q)\,\Pi(q)\,V^{RPA}(q)$$

$$V^{RPA}(q) = \frac{V(q)}{1 - V(q)\,\Pi(q)} = \frac{V(q)}{\varepsilon(q)}$$

$$\boxed{\varepsilon = \text{dielectric fct.}}$$

$$D(q,\omega) = \varepsilon(q,\omega)\, E(q,\omega)$$

$$D = \triangledown\, V^{RPA}$$

$$V^{RPA} = \frac{1}{\varepsilon} V$$

$$E = \triangledown\, V$$

$$\varepsilon = 1 + 4\pi \chi$$

$$\chi = \frac{1}{4\pi} V(q)\, \Pi(q)$$

$$\Pi(q) \xrightarrow{\text{low } \omega} -\rho(\mu) + O\left(\frac{\omega}{v_F q}\right)$$

$$V^{RPA}_{(q)} = \frac{U(q)}{1+U(q)\,\rho(\mu)} = \frac{e^2}{q^2 + \underbrace{\rho e^2}_{\lambda^{-2}}}$$

$$\Big\downarrow \text{FT}$$

$$V^{RPA}_{(r)} = \frac{e^2}{|r|}\, e^{-|r|/\lambda}$$

Thomas-Fermi
Screening length