The Ising model

Spins $s_i = \pm 1$

$$H = \frac{1}{2}\sum_{i,j} J_{ij} s_i s_j - B\sum_i s_i$$

Easiest: $J_{ij} = \begin{cases} J & \text{if } i,j \text{ nearest neighbors} \\ 0 & \text{otherwise} \end{cases}$

If $J<0$: ferromagnetic

$J>0$: antiferromag.

Soln in 1d: transfer matrix

$$H = \sum_{i=0}^{N-1} H_{s_i s_{i+1}} \qquad H_{s_i s_{i+1}} = -\overset{-J}{\overset{\downarrow}{\varepsilon}} s_i s_{i+1} + \frac{B}{2}(s_i + s_{i+1})$$

Part^n f^n $Z = \mathrm{Tr}\, e^{-\beta H}$

because $Z = \sum_{\{s\}} \prod_{i=0}^{N-1} e^{-\beta H_{s_i s_{i+1}}} = \mathrm{Tr}\, T^N$

$$Z = \sum_{\text{all states}} e^{-\beta E_{\text{state}}}$$

Idea of transfer matrix:

Consider $e^{-\beta H_{s_i s_{i+1}}}$ as a matrix

$$\begin{array}{ccc} s_i=+1 & s_i=-1 & \\ \left(\begin{array}{c} e^{\beta(\varepsilon+B)} \\ \\ e^{-\beta\varepsilon} \end{array}\right. & \left.\begin{array}{c} e^{-\beta\varepsilon} \\ \\ e^{\beta(\varepsilon-B)} \end{array}\right) & \begin{array}{l} s_{i+1}=+1 \\ \\ s_{i+1}=-1 \end{array} \end{array} \equiv T$$

Multiplying for many successive i's:

$$\cdots \quad \overset{s_1 \to}{\begin{pmatrix} & \\ & \end{pmatrix}} \Big\downarrow_{s_2} \overset{s_0 \to}{\begin{pmatrix} & \\ & \end{pmatrix}} \Big\downarrow_{s_1}$$

So we get $Z = T_r T^N$

T can be diagonalized

then $T = \begin{pmatrix} \lambda_0 & 0 \\ 0 & \lambda_1 \end{pmatrix}$

& $T^N = \begin{pmatrix} \lambda_0^N & 0 \\ 0 & \lambda_1^N \end{pmatrix}$

so $Z = \lambda_0^N + \lambda_1^N$

Explicit calculation (ex.):

2 eigenvalues; keeping largest one,

$$\lambda_0 = e^{\beta \varepsilon}\left[\cosh \beta B + \sqrt{\cosh^2 \beta B - (1 - e^{-4\beta \varepsilon})}\right]$$

so that in the limit $N \to \infty$,

$$\lim_{N\to\infty} \frac{1}{N} \ln Z = \lim \frac{1}{N} \ln \operatorname{Tr} T^N$$

$$= \mathcal{L} \frac{1}{N} \ln \lambda_0^N + \underbrace{\lambda_1^N}_{\text{neglect}} = \ln \lambda_0$$

Free energy:

$$F = -T \ln Z$$

$$= -\varepsilon - \frac{1}{\beta} \ln [\ \ ]$$

To compute the average magnetiz^n:

$$\langle s \rangle = -\left(\frac{\partial F}{\partial B}\right)\bigg|_T = \underbrace{\sinh \beta B}_{\dots}$$

so $\lim_{B\to 0} \langle s \rangle = 0$ No spontaneous magnet^n.

High-temperature expansions

$$Z = \sum_{\{S_i\}} e^{-\beta H(\{S_i\})} = \sum_{\{S_i\}} \sum_{n=0}^{\infty} \frac{[-\beta H(\{S_i\})]^n}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \sum_{\{S_i\}} [\beta H(\{S_i\})]^n$$

For simplicity: put field $= 0$. $Z = \sum_{\{S_i\}} \prod_{\langle i,j \rangle} e^{\beta \varepsilon S_i S_j}$

"nearest neighbor pair"

Use $e^{\pm A} = \cosh A \pm \sinh A = \cosh A [1 \pm \tanh A]$

$v \equiv \tanh \beta\varepsilon$

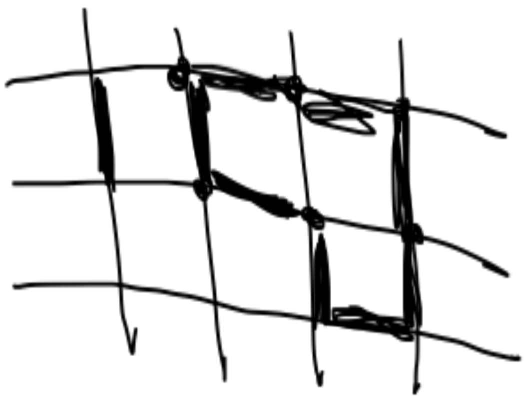
so $Z = \sum_{\{S_i\}} \prod_{\langle ij \rangle} \cosh \beta\varepsilon [1 + S_i S_j \tanh(\beta\varepsilon)] = [\cosh \beta\varepsilon]^{\frac{Nz}{2}} \sum_{\{S_i\}} \prod_{\langle i,j \rangle} [1 + S_i S_j v]$

The summand is a polynomial in v:

$$\prod_{\langle i,j\rangle}\left[1+S_i S_j v\right] = 1 + v\sum_{\langle i,j\rangle} S_i S_j + v^2 \sum_{\substack{\langle i,j\rangle \\ \neq \langle k,l\rangle}} S_i S_j S_k S_l + v^3 \ldots$$

Remember the $\sum_{\{S_i\}}$ $\qquad v\sum_{S_i} S_i S_j$

Picture: on the lattice, associate $S_i S_j$ to a link between sites i & j.



So: nice "version" of part. f.:

$$Z = [\cosh \beta\varepsilon]^{\frac{Nz}{2}}\, 2^N \sum_{l=0}^{\infty} g(l)\, v^l$$

$g(l)$ = # loops of length l

Let's count loops on square lattice

Lattice of N sites, $N = L^2$

Definition: - a closed path: an arrowed set of links starting & finishing at a point.

- a closed path is <u>connected</u> if it's composed of a single set of links touching each other

Define the # of closed connected paths of length l as $h(l)$

For convenience, define

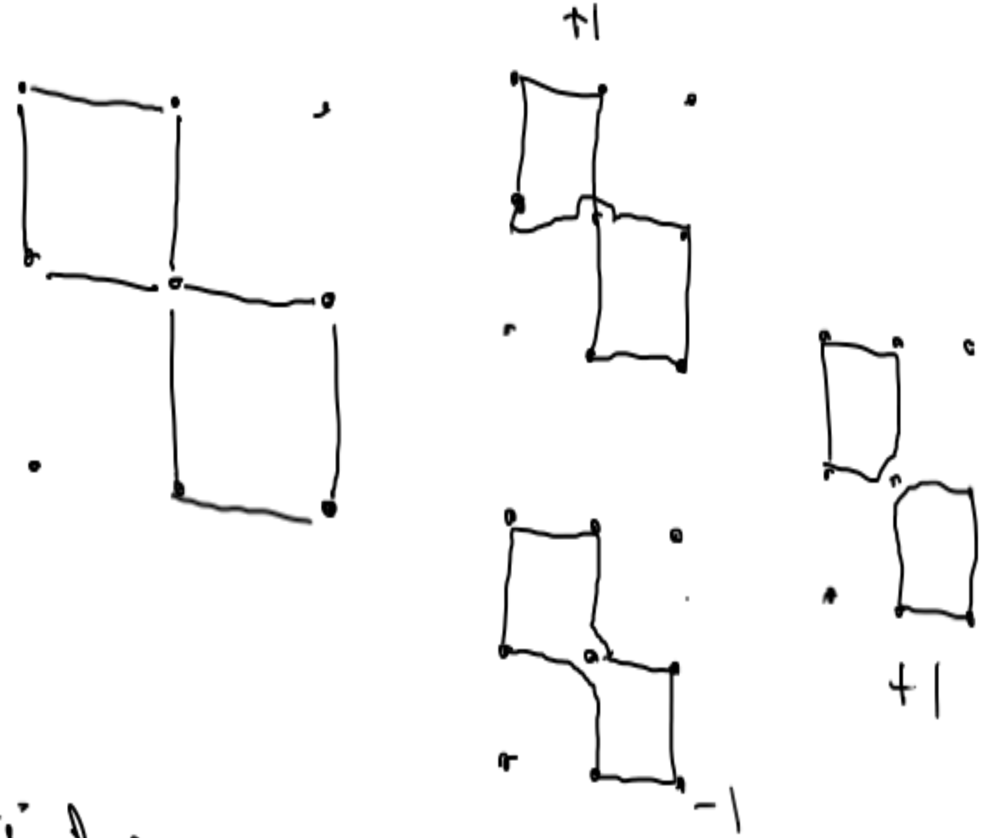
$$D(l) \equiv \frac{1}{2l} h(l)$$

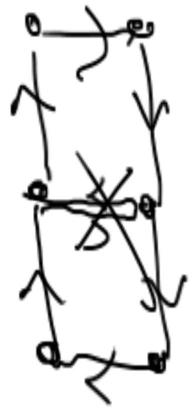
What we expect: s.t. like

$$g(l) \overset{?}{=} \sum_{n=1}^{l} \frac{1}{n!} \sum_{\substack{l_1,\dots,l_n \\ \sum l_i = l}} D(l_1) \dots D(l_n)$$

For a left turn, $e^{i\pi/4}$

right ", $e^{-i\pi/4}$





Critical error: paths overcount loops.

Trick to correct for this:
include phases in our paths.

Let's try to count paths with matrices.

Let $\underset{\sim}{M}$ be a matrix with elements $M_{\underset{\sim}{i}\underset{\sim}{j}}$ is $\neq 0$ if $\underset{\sim}{i}$ & $\underset{\sim}{j}$ can be linked by a single link.

By extension, $\left(M^l\right)_{\underset{\sim}{i}\underset{\sim}{j}}$ will be $\neq 0$ if $\underset{\sim}{i}$ & $\underset{\sim}{j}$ can be linked by l segments

From rules of matrix multiplication, if we set $M_{\underset{\sim}{i}\underset{\sim}{j}} = 1$ if $\underset{\sim}{i}$ & $\underset{\sim}{j}$ are nearest neighbors, then $\left(M^l\right)_{\underset{\sim}{i}\underset{\sim}{j}}$ is the nr of paths linking $\underset{\sim}{i}$ & $\underset{\sim}{j}$ by l segments

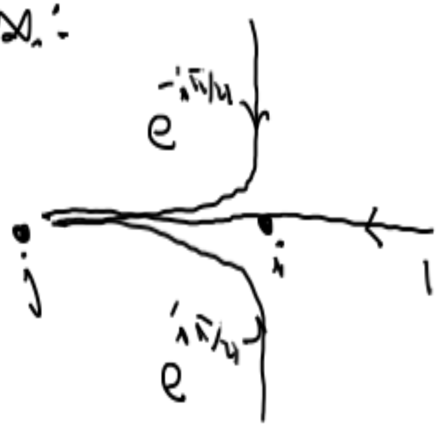
$$\left(M^2\right)_{\underset{\sim}{i}\underset{\sim}{j}} = \sum_{\underset{\sim}{k}} M_{\underset{\sim}{i}\underset{\sim}{k}} M_{\underset{\sim}{k}\underset{\sim}{j}}$$

Not done yet: include phases.

Trick: write each element of M as a 4×4 matrix with indices determined by incoming & outgoing directions.

$$M_{\underset{\sim}{i}\underset{\sim}{j}} : \quad \underset{\sim}{m}^{\alpha\beta}_{\underset{\sim}{i}\underset{\sim}{j}} \qquad \begin{array}{ccccc} \alpha,\beta = & 0 & 1 & 2 & 3 \\ & E & N & W & S \end{array}$$

Ex.:



$$\begin{array}{c} \text{incoming} \downarrow \end{array} \overset{\text{outgoing} \rightarrow}{\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & e^{i\pi/4} & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & e^{-i\pi/4} & 0 \end{pmatrix}} = m_{(1,0)}$$

Really: $m_{\underset{\sim}{i}-\underset{\sim}{j}}$

Now: $\sum_{\alpha} \left(M^{l}\right)^{\alpha\alpha}_{\underset{\sim}{i}\underset{\sim}{i}}$ counts the closed paths going from $\underset{\sim}{i}$ to $\underset{\sim}{i}$ in l steps

Can now count the nr of loops:

$$D(l) = -\frac{1}{2l}\operatorname{Tr} M^{l} = -\frac{1}{2l}\sum_{\underset{\sim}{i}} \operatorname{Tr}_{4}\, m^{l}_{\underset{\sim}{i}\underset{\sim}{i}}$$

Back to Z:

$$Z = \left[\cosh\beta\varepsilon\right]^{\frac{Nz}{2}} 2^N \sum_l g(l)\, v^l$$

$$= [\;]^{\frac{Nz}{2}} 2^N \left\{ 1 + \sum_{l=1}^{\infty} v^l \sum_{m=1}^{l} \frac{1}{m!} \sum_{\substack{l_1, l_2, \ldots l_m \\ l_1 + \ldots + l_m = l}} D(l_1) \ldots D(l_m) \right\}$$

$\underset{\sim}{M}$ can be diagonalized. Using eigenvalues,

$$D(l) = \frac{-1}{2l} \operatorname{Tr} M^l = \frac{-1}{2l} \sum_{\alpha=0}^{4N-1} \lambda_\alpha^l$$

so $Z = [\;]^{\frac{Nz}{2}} 2^N \exp\left\{ -\sum_{\alpha=0}^{4N-1} \sum_{l=1}^{\infty} \frac{\lambda_\alpha^l v^l}{2l} \right\}$

Recognizing series for log,

$$-\ln(1-x) = \sum_{n=1}^{\infty} \frac{x^n}{n},$$

get $Z = \left[\cosh\beta\varepsilon\right]^{\frac{Nz}{2}} 2^N \prod_\alpha \left[1 - v\lambda_\alpha\right]^{1/2}$

Bit more work (see notes) gives

$$Z = \left[\cosh\beta\varepsilon\right]^{\frac{Nz}{2}} 2^N \prod_q \left[(1+v^2)^2 - 2v(1-v^2)\left[\cos q_1 + \cos q_2\right]\right]^{1/2}$$

is the exact Z for Ising in 2d.