Random walks

$d$-dim$^{l}$ $\square$ lattice $\mathcal{L}_a$

coordination nr: $c_{\mathcal{L}_a} = 2d$

2 rules:

rule 1: at each $\delta t$, walker takes one step

rule 2: the direction of the step is uniformly distributed

Notation: $(t_1 = t_0 + n\delta t)$

$$P_{\underset{\sim}{r}_1, t_1 | \underset{\sim}{r}_0, t_0}$$

choose initial cond$^{n}$

$$P_{\underset{\sim}{r}_1, t_0 | \underset{\sim}{r}_0, t_0} = \delta_{\underset{\sim}{r}_1, \underset{\sim}{r}_0}$$

Must have $\forall\, \underset{\sim}{r}_1, t_1$:

$$0 \leq P_{\underset{\sim}{r}_1, t_1 | \underset{\sim}{r}_0, t_0} \leq 1$$

& $\forall t_1$,

$$\sum_{\underset{\sim}{r}_1} P_{\underset{\sim}{r}_1, t_1 | \underset{\sim}{r}_0, t_0} = 1$$

Also: $P_{\underset{\sim}{x}_1, t_1 | \underset{\sim}{x}_0, t_0} = 0$ if $|\underset{\sim}{x}_1 - \underset{\sim}{x}_0| > a \dfrac{t_1 - t_0}{\delta t}$

Discrete time-evol$^{\underline{n}}$ eq$^{\underline{n}}$:

$$P_{\underset{\sim}{x}_1, t_1 + \delta t | \underset{\sim}{x}_0, t_0} = \frac{1}{2d} \sum_{\mu=1}^{d} \sum_{\sigma=\pm 1} P_{\underset{\sim}{x}_1 + a\sigma \hat{\underset{\sim}{n}}_\mu, t_1 | \underset{\sim}{x}_0, t_0}$$

Discrete Laplacian:

$$\nabla_a^2 f_{\underset{\sim}{x}} = \frac{1}{a^2} \sum_{\mu=1}^{d} \left[ f_{\underset{\sim}{x} + a\hat{n}_\mu} + f_{\underset{\sim}{x} - a\hat{n}_\mu} - 2 f_{\underset{\sim}{x}} \right]$$

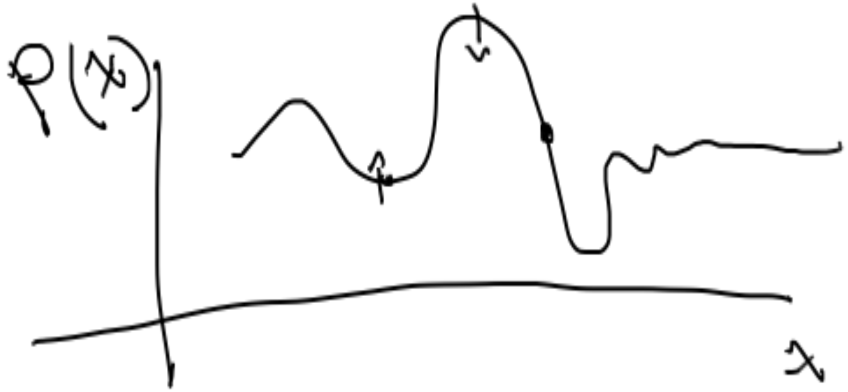
$$\rightarrow P_{\underset{\sim}{x}_1, t_1 + \delta t | \underset{\sim}{x}_0, t_0} - P_{\underset{\sim}{x}_1, t_1 | \underset{\sim}{x}_0, t_0} = \frac{a^3}{2d} \nabla_a^2 P_{\underset{\sim}{x}_1, t_1 | \underset{\sim}{x}_0, t_0}$$

Discrete version of a famous eq$^{n}$: $\left[\frac{\partial}{\partial t} - D\nabla^2\right] P(\underset{\sim}{x}, t) = 0$

→ diffusion eq$^{n}$

diffusion constant

Graphics:




For our walker: $D = \lim\limits_{\substack{a \to 0 \\ \delta t \to 0}} \frac{a^2}{2d\,\delta t}$

Go back to discrete $r$:

Solution: Fourier series

Convention:

$$f_{\underset{\sim}{r}} = a^d \int_{-\pi/a}^{\pi/a} \frac{d^d \underset{\sim}{k}}{(2\pi)^d} e^{i \underset{\sim}{k} \cdot \underset{\sim}{r}} f_{(\underset{\sim}{k})}$$

$$f_{\underset{\sim}{k}} = \sum_{\underset{\sim}{r}} e^{-i \underset{\sim}{k} \cdot \underset{\sim}{r}} f_{\underset{\sim}{r}}$$

Initial cond$^n$

$$P_{\underset{\sim}{r}_1, t_0 | \underset{\sim}{r}_0, t_0} = \delta_{\underset{\sim}{r}_1, \underset{\sim}{r}_0}$$

$$\text{FT} \to \quad P_{\underset{\sim}{k}, t_0 | \underset{\sim}{r}_0, t_0} = e^{-i \underset{\sim}{k} \cdot \underset{\sim}{r}_0}$$

Time evol$^n$ eq$^n$:

$$P_{\underset{\sim}{k}, t_1 + \delta t | \underset{\sim}{r}_0, t_0} = \left[ \frac{1}{d} \sum_{\mu=1}^{d} \cos(k^\mu a) \right] P_{\underset{\sim}{k}, t_1 | \underset{\sim}{r}_0, t_0}$$

$$\to P_{\underset{\sim}{k}, t_1 + n\delta t | \underset{\sim}{r}_0, t_0} = \left[ \quad \right]^n P_{\underset{\sim}{k}, t_1 | \underset{\sim}{r}_0, t_0}$$

Can immediately write down the solⁿ: go back to real space

$$\boxed{P_{\underset{\sim}{r}_1, t_1 | \underset{\sim}{r}_0, t_0} = a^d \int_{-\pi/a}^{\pi/a} \frac{d^d \underset{\sim}{k}}{(2\pi)^d} e^{i \underset{\sim}{k} \cdot (\underset{\sim}{r}_1 - \underset{\sim}{r}_0)} \left[ \frac{1}{d} \sum_{\mu=1}^{d} \cos(k^\mu a) \right]^{\frac{t_1 - t_0}{\delta t}}}$$

→ NO APPROXIMATIONS

$$\lim_{N\to\infty} \left( 1 + \frac{\beta}{N} \right)^N$$

Continuum limit: $a \to 0$, $\delta t \to 0$

Approx: $$\left[ \frac{1}{d} \sum_{\mu=1}^{d} \cos k^\mu a \right]^{\frac{t_1 - t_0}{\delta t}} = \left[ 1 - \frac{a^2}{2d} \underset{\sim}{k}^2 + \ldots \right]^{\frac{t_1 - t_0}{\delta t}} \simeq e^{-(t_1 - t_0) \underbrace{\frac{a^2}{2d\delta t}}_{D} k^2}$$

Because of continuum limit: use proba density

$$P(\underset{\sim}{r}_1, t_1 | \underset{\sim}{r}_0, t_0) \equiv \lim \frac{1}{a^d} P_{\underset{\sim}{r}_1 \ldots} = \int_{-\infty}^{\infty} \frac{d^d \underset{\sim}{k}}{(2\pi)^d} e^{-(t_1 - t_0) D \underset{\sim}{k}^2 + i \underset{\sim}{k} \cdot (\underset{\sim}{r}_1 - \underset{\sim}{r}_0)}$$

Miracle:

$$\boxed{P(\underset{\sim}{r}_1, t_1 | \underset{\sim}{r}_0, t_0) = \frac{1}{[4\pi D(t_1 - t_0)]^{\frac{d}{2}}} e^{-\frac{|\underset{\sim}{r}_1 - \underset{\sim}{r}_0|^2}{4D(t_1 - t_0)}}}$$

Gaussian integral!

so $|\underset{\sim}{r}_1 - \underset{\sim}{r}_0| \sim (t_1 - t_0)^\nu \quad \nu = 1/2$

Useful property: composition rule $\quad t_2 \geq t_1 \geq t_0$

$$\int d\underset{\sim}{r}_1 \, P(\underset{\sim}{r}_2, t_2 | \underset{\sim}{r}_1, t_1) P(\underset{\sim}{r}_1, t_1 | \underset{\sim}{r}_0, t_0) = P(\underset{\sim}{r}_2, t_2 | \underset{\sim}{r}_0, t_0)$$

Further Q: how much time does our walker spend on a given site?

$$\sum_{n=0}^{\infty} \gamma^n = \frac{1}{1-\gamma} \quad |\gamma|<1$$

Discrete:

$$\sum_{n=0}^{\infty} P_{\underset{\sim}{r}_1, t_0+n\delta t | \underset{\sim}{r}_0, t_0} = a^d \int \frac{d^d \underset{\sim}{k}}{(2\pi)^d} \frac{e^{i \underset{\sim}{k} \cdot (r_1 - r_0)}}{1 - \frac{1}{d}\sum_1^d \cos k^\mu a} \equiv \mathcal{G}_{r_1 - r_0}$$

"Green's fn": Obeys: $-\nabla_a^2 \mathcal{G}_{\underset{\sim}{r}_1 - \underset{\sim}{r}_0} = \frac{2d}{a^2} \delta_{\underset{\sim}{r}_1, \underset{\sim}{r}_0}$

Continuum:

$$\sim \int \underbrace{d^d \underset{\sim}{k}}_{\sim k^{d-1}} \frac{e^{i \underset{\sim}{k} \cdot \underset{\sim}{r}}}{\underset{\sim}{k}^2}$$

$d = 3, 4, \ldots$ OK

$d = 2 \longrightarrow$ log divergence

$d = 1 \longrightarrow$ bad divergence

Trick to make sense of this: subtraction

$$G_{\underline{n}_i - \underline{n}_0} - G_0 \equiv G^S_{\underline{n}_i - \underline{n}_0} = \dots \int d^d k \; \frac{e^{i \underline{k} \cdot \underline{n}} - 1}{\dots}$$

Trick #2: regularization

→ add rule: at each time step, particle disappears with proba $\eta$.