1.3 Path integral

$$P_{\underset{\sim}{r}_1, t_1 | \underset{\sim}{r}_0, t_0} = \frac{\#\text{ paths from } r_0, t_0 \text{ to } r_1, t_1}{\#\text{ paths}}$$

Scaling limit.

$$P(\underset{\sim}{r}_f, t_f | \underset{\sim}{r}_i, t_i) = \frac{e^{-|\underset{\sim}{r}_f - \underset{\sim}{r}_i|^2 / 4D(t_f - t_i)}}{4\pi D (t_f - t_i)^{d/2}}$$

composition rule: N time slices $t_f - t_i = \Delta t N$

$$P(\underset{\sim}{r}_f, t_f | \underset{\sim}{r}_i, t_i) = \int \prod_{n=1}^{N-1} d^d \underset{\sim}{r}_n \; P(r_f, t_f | r_{N-1}, t_{N-1}) \cdots P(r_1, t_1 | r_i, t_i)$$

Simplify the prob for stepping from n to $n+1$: $r_{n+1} - r_n$

$$P(r_{n+1}, t_{n+1} | r_n, t_n) \xrightarrow[t_{n+1}-t_n = \Delta t \to 0]{} \frac{1}{[4\pi D \Delta t]^{d/2}} \exp\left\{ -\frac{\Delta t}{4D} \left| \frac{\Delta r(t_n)}{\Delta t} \right|^2 \right.$$

& use $\dfrac{\Delta r}{\Delta t} \to \dfrac{dr(t)}{dt}$

$P = \int \mathcal{D}r \, e^{-S[r]}$

Our full prob: as a path integral

$$P(r_s, t_s | r_i, t_i) = \int_{\substack{r(t_i) = r_i \\ r(t_s) = r_s}} \mathcal{D} r(t) \exp\left\{ -\frac{1}{4D} \int_{t_i}^{t_s} dt \left| \frac{dr(t)}{dt} \right|^2 \right\}$$

where the P.I. measure $\mathcal{D}r$ is defined as

$$\int_{\substack{r(t_i)=r_i \\ r(t_s)=r_s}} \mathcal{D}r(t)\, F[r(t)] = \lim_{N\to\infty} \left[\frac{N}{4\pi D(t_s-t_i)}\right]^{Nd/2} \int \prod_{m=1}^{N-1} dr_m\, F(\{r_m\})$$